
误差理论与测量平差基础

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函数模型与引入参数个数的关系

引入参数情况 → 函数模型形式 → 平差方法类型 → 平差结果不变

引入参数 $\hat{X}$ 个数 u	函数模型			平差方法 类型
	个数	形式	名称	
$u=0$	$c=r$	$F_{r1}(\hat{L}_{n1}) = 0$	条件方程	条件平差(4)
$u < t$ 独立	$c=r+u$	$F_{c1}(\hat{L}_{n1}, \hat{X}_{u1}) = 0$	附有参数的条件方程	附有参数的条件平差(5)
$u=t$ 独立	$c=r+t=n$	$\hat{L}_{n1} = F_{n1}(\hat{X}_{t1})$	误差方程	间接平差(6)
$u > t$ 其中 $s=u-t$ 不独立	$c=r+u$ $=n+s$	$\hat{L}_{n1} = F_{n1}(\hat{X}_{u1})$ $\Phi_{s1}(\hat{X}_{u1}) = 0$	误差方程+限制条件方程	附有限制条件的间接平差(7)

4种平差方法的函数模型变换 $(\hat{L}, \hat{X}) \rightarrow (\mathbf{V}, \hat{x})$

作变量拆分: $\hat{L}_{n1} = L_{n1} + V_{n1}$ $\hat{X}_{u1} = \hat{X}_{u1}^0 + \hat{x}_{u1}$

方法平差	引入参数 $\hat{X}$ 个数 u	$\hat{L}, \hat{X}$ 的函数模型			$\mathbf{V}, \hat{x}$ 的函数模型
		名称	非线性	线性	
条件平差	$u=0$	条件方程	$F_{r1}(\hat{L}_{n1}) = 0$	$A_{rn\ n1} \hat{L}_{n1} + A_{r1} = 0$	$A_{rn\ n1} V_{n1} + W_{r1} = 0$ $W_{r1} = F(L) = AL + A_0$
附参条件平差	$u < t$ 独立 $\hat{X}_{u1}$	附参条件方程	$F_{c1}(\hat{L}_{n1}, \hat{X}_{u1}) = 0$	$A_{cn\ n1} \hat{L}_{n1} + B_{cu\ u1} \hat{X}_{u1} + A_{c1} = 0$	$A_{cn\ n1} V_{n1} + B_{cu\ u1} \hat{x}_{u1} + W_{c1} = 0$ $W_{c1} = F(L, X^0) = AL + BX^0 + A_0$
间接平差	$u=t$ 独立 $\hat{X}_{t1}$	误差方程	$\hat{L}_{n1} = F_{n1}(\hat{X}_{t1})$	$\hat{L}_{n1} = B_{nt\ t1} \hat{X}_{t1} + d_{n1}$	$V_{n1} = B_{nt\ t1} \hat{x}_{t1} - l_{n1}$ $l = L - F(X)$ $= L - (BX + d)$
附限间接平差	$u > t$ 其中 $s = u - t$ 不独立 $\hat{X}_{u1}$	误差方程+限制条件方程	$\hat{L}_{n1} = F_{n1}(\hat{X}_{u1})$ $\Phi_{s1}(\hat{X}_{u1}) = 0$	$\hat{L}_{n1} = B_{nu\ u1} \hat{X}_{u1} + d_{n1}$ $C_{su\ u1} \hat{X}_{u1} - C_{s1} X_{s1}^0 + W_{s1}^x = 0$	$V_{n1} = B_{nu\ u1} \hat{x}_{u1} - l_{n1}$ $C_{su\ u1} \hat{x}_{u1} + W_{s1}^x = 0$ $W_{s1}^x = \Phi_s(X^0)$

$$A = \left(\frac{\partial F}{\partial \hat{L}} \right)_0 \quad B = \left(\frac{\partial F}{\partial \hat{X}} \right)_0 \quad C = \left(\frac{\partial \Phi_s}{\partial \hat{X}} \right)_0$$

第四章 条件平差

§4-1 条件平差原理

§4-2 平差结果精度评价

§4-3 条件方程类型

§4-4 条件平差实例

§4-1 条件平差原理

平差的任务：求 $\hat{L}$ 、 $\hat{X}$

引入参数个数 不引入 $u=0$	函数模型个数 $c=r$	函数模型形式 $F_{r1}(\hat{L})=0$	函数模型名称 条件方程	平差方法名称 条件平差(4)
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1、平差原理(怎样由 L 求 $\hat{L}$?)

平差问题中，不引入未知参数， r 个多余观测，产生 r 个函数模型-条件方程：

$$\begin{cases} \text{非线性: } F_{r1}(\hat{L})=0 \\ \text{线性: } A_{rm\ n1} \hat{L} + A_{r1} = 0 \end{cases} \xrightarrow{\hat{L} = L + V} A_{rm\ n1} V + W_{r1} = 0$$

改正数条件方程

$$F_{r1}(\hat{L}) \approx F_{r1}(L) + \left(\frac{\partial F}{\partial \hat{L}}\right)_o (\hat{L} - L) = A_{rm\ n1} V + W_{r1}$$

$$\text{系数矩阵: } A_{rm} = \left(\frac{\partial F_{r1}}{\partial \hat{L}_{n1}}\right)_o$$

$$\text{闭合差: } W_{r1} = F(L) = AL + A_0$$

任务转化为由 W 求 V 。但 r 个条件方程的 n 个 V 存在多解，怎样的 V 更好？

1、平差原理(由L求 $\hat{L}$?)

(几何关系)

$$\left. \begin{array}{l} \text{函数模型: } \underset{rn \ n1}{A} \underset{r1}{V} + \underset{r1}{W} = \underset{r1}{0} \\ \text{平差原则: } \underset{1n}{V}^T \underset{nn \ n1}{P} \underset{n1}{V} = \min \end{array} \right\} \Rightarrow \underset{n1}{V} = ?$$

$$\text{随机模型: } \underset{n1}{D} = \underset{0}{\sigma_0^2} \underset{n1}{P}^{-1} = \underset{0}{\sigma_0^2} \underset{n1}{Q}$$

$$\text{为此构建一目标函数: } \underset{11}{\Phi}(\underset{n1}{V}) = \underset{1 \ 1}{V}^T \underset{1 \ 1}{P} \underset{1 \ 1}{V} - 2 \underset{1r}{K}^T (\underset{r1}{A} \underset{n1}{V} + \underset{r1}{W})$$

$$\text{待定的联系数矩阵 } \underset{n1}{K} = (\underset{1}{k_1}, \underset{2}{k_2} \text{ L } \underset{r}{k_r})^T$$

现求合适的V, 使 $\Phi(V)=\min$ 。必要条件是:

$$\frac{\partial}{\partial \underset{n1}{V}} \underset{11}{\Phi}(\underset{n1}{V}) = 2 \underset{n1}{V}^T \underset{n1}{P} - 2 \underset{n1}{K}^T \underset{n1}{A} = 0$$

$$\underset{n1}{V} = \underset{n1}{P}^{-1} \underset{n1}{A}^T \underset{n1}{K} = \underset{n1}{Q} \underset{n1}{A}^T \underset{n1}{K} \quad (\text{极值关系})$$

这是一个条件极值问题:
在 $AV + W = 0$ 的条件下,
求 $V^T P V = \min$ 的极值点V

$$\underset{n1}{V} = \underset{n1}{V}(t) = [\underset{1}{v_1}(t), \underset{2}{v_2}(t) \text{ L } \underset{n}{v_n}(t)]^T$$

$$\text{因为 } \frac{d(\underset{11}{V}^T \underset{1 \ 1}{P} \underset{1 \ 1}{V})}{dt} = \frac{\partial(\underset{11}{V}^T \underset{1 \ 1}{P} \underset{1 \ 1}{V})}{\partial \underset{1n}{V}} \frac{d\underset{n1}{V}}{dt}$$

$$\begin{aligned} \text{又 } \frac{d(\underset{11}{V}^T \underset{1 \ 1}{P} \underset{1 \ 1}{V})}{dt} &= \frac{d\underset{1n}{V}^T}{dt} (\underset{nn \ n1}{P} \underset{n1}{V}) + \underset{1n}{V}^T \frac{d(\underset{n1}{P} \underset{n1}{V})}{dt} \\ &= \left[\frac{d(\underset{1n}{V}^T)}{dt} (\underset{n1}{P} \underset{n1}{V}) \right]^T + \underset{1n}{V}^T \underset{n1}{P} \frac{d\underset{n1}{V}}{dt} \\ &= 2 \underset{1n}{V}^T \underset{n1}{P} \frac{d\underset{n1}{V}}{dt} \end{aligned}$$

$$\text{所以 } \frac{\partial(\underset{11}{V}^T \underset{1 \ 1}{P} \underset{1 \ 1}{V})}{\partial \underset{n1}{V}} = 2 \underset{1n}{V}^T \underset{n1}{P}$$

(几何关系) $F_{r1}(\hat{L}) = 0 \rightarrow A_{m\ n1} V + W_{r1} = 0$

(极值关系) $V^T P V = \min \rightarrow V_{n1} = Q A^T K_{r1}$

⇒ 求V、K

V代入，得法方程 $N_{aa} K_{r1} + W_{r1} = 0$

$$N_{aa} = A_{m\ n1} Q A_{n1\ nr}^T \quad R(N_{aa}) = r, \quad N_{aa} \text{满秩、可逆}$$

解法方程求K $K_{r1} = -N_{aa}^{-1} W_{r1}$

代入得改正数V、平差值 $\hat{L}$

$$V_{n1} = Q A^T K_{r1} = -Q A^T N_{aa}^{-1} W_{r1}$$

$$\hat{L}_{n1} = L + V = L - Q A^T N_{aa}^{-1} W_{r1}$$

实现了平差求 $\hat{L}$ 的任务！

1、平差原理（矩阵式-代数式）

条件方程

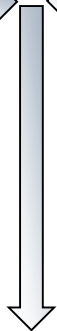
$$F_{r1}(\hat{L}) = 0$$

$$\begin{cases} F_1(\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n) = 0 \\ F_2(\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n) = 0 \\ \vdots \\ F_r(\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n) = 0 \end{cases}$$



$$A_{m \ n1} \hat{L} + A_{r1} = 0$$

$$\begin{cases} a_{11}\hat{L}_1 + a_{12}\hat{L}_2 + \dots + a_{1n}\hat{L}_n + A_{10} = 0 \\ a_{21}\hat{L}_1 + a_{22}\hat{L}_2 + \dots + a_{2n}\hat{L}_n + A_{20} = 0 \\ \vdots \\ a_{r1}\hat{L}_1 + a_{r2}\hat{L}_2 + \dots + a_{rn}\hat{L}_n + A_{r0} = 0 \end{cases}$$



改正数条件方程

$$AV + W = 0$$

$$\begin{cases} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n + w_1 = 0 \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n + w_2 = 0 \\ \vdots \\ a_{r1}v_1 + a_{r2}v_2 + \dots + a_{rn}v_n + w_r = 0 \end{cases}$$

$$A = \left(\frac{\partial F_{r1}}{\partial \hat{L}_{n1}} \right)_0 = \begin{pmatrix} \frac{\partial F_1}{\partial \hat{L}_1} & \frac{\partial F_1}{\partial \hat{L}_2} & \dots & \frac{\partial F_1}{\partial \hat{L}_n} \\ \frac{\partial F_2}{\partial \hat{L}_1} & \frac{\partial F_2}{\partial \hat{L}_2} & \dots & \frac{\partial F_2}{\partial \hat{L}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_r}{\partial \hat{L}_1} & \frac{\partial F_r}{\partial \hat{L}_2} & \dots & \frac{\partial F_r}{\partial \hat{L}_n} \end{pmatrix}_0 = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rn} \end{pmatrix}$$

$$W_{r1} = F_{r1}(L) = \begin{pmatrix} F_1(L) \\ F_2(L) \\ \vdots \\ F_r(L) \end{pmatrix} = \begin{pmatrix} a_{11}L_1 + a_{12}L_2 + \dots + a_{1n}L_n + A_{10} \\ a_{21}L_1 + a_{22}L_2 + \dots + a_{2n}L_n + A_{20} \\ \vdots \\ a_{r1}L_1 + a_{r2}L_2 + \dots + a_{rn}L_n + A_{r0} \end{pmatrix}$$

1、平差原理（矩阵式-代数式）

法方程 $N_{aa} \mathbf{K}_{r1} + \mathbf{W}_{r1} = 0$

改正数

不相关观测

$$\begin{cases} \sum_{i=1}^n \frac{a_{1i} a_{1i}}{p_i} \mathbf{k}_1 + \sum_{i=1}^n \frac{a_{1i} a_{2i}}{p_i} \mathbf{k}_2 \mathbf{L} + \sum_{i=1}^n \frac{a_{1i} a_{ri}}{p_i} \mathbf{k}_r + w_1 = 0 \\ \sum_{i=1}^n \frac{a_{2i} a_{1i}}{p_i} \mathbf{k}_1 + \sum_{i=1}^n \frac{a_{2i} a_{2i}}{p_i} \mathbf{k}_2 \mathbf{L} + \sum_{i=1}^n \frac{a_{2i} a_{ri}}{p_i} \mathbf{k}_r + w_2 = 0 \\ \vdots \\ \sum_{i=1}^n \frac{a_{ri} a_{1i}}{p_i} \mathbf{k}_1 + \sum_{i=1}^n \frac{a_{ri} a_{2i}}{p_i} \mathbf{k}_2 \mathbf{L} + \sum_{i=1}^n \frac{a_{ri} a_{ri}}{p_i} \mathbf{k}_r + w_r = 0 \end{cases}$$

$$\mathbf{V}_{n1} = \mathbf{Q} \mathbf{A}^T \mathbf{K} = -\mathbf{Q} \mathbf{A}^T \mathbf{N}_{aa}^{-1} \mathbf{W}$$

$$v_i = \frac{1}{p_i} \sum_{j=1}^r a_{ji} k_j$$

L的平差值

$$\hat{\mathbf{L}}_{n1} = \mathbf{L} + \mathbf{V} = \mathbf{L} - \mathbf{Q} \mathbf{A}^T \mathbf{N}_{aa}^{-1} \mathbf{W}$$

$$N_{aa} = \mathbf{A} \mathbf{Q} \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{L} & \mathbf{L} & & \\ a_{r1} & a_{r2} & \mathbf{L} & a_{rn} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{L} & \mathbf{L} & & \\ a_{r1} & a_{r2} & \mathbf{L} & a_{rn} \end{pmatrix}^T$$

$$\underline{\underline{\text{不相关观测}}} = \begin{pmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{L} & \mathbf{L} & & \\ a_{r1} & a_{r2} & \mathbf{L} & a_{rn} \end{pmatrix} \begin{pmatrix} 1/p_1 & & & \\ & 1/p_2 & & \\ & & \mathbf{O} & \\ & & & 1/p_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{L} & \mathbf{L} & & \\ a_{r1} & a_{r2} & \mathbf{L} & a_{rn} \end{pmatrix}^T$$

2、平差计算步骤与示例

① 建立函数模型。

立列 r 个条件方程 $F(\hat{L})=0$ ，线性化变换为 $AV+W=0$ 。

② 建立随机模型。取 σ_0^2 ，确定观测值的协因数阵 Q 。

③ 建立法方程。 $N_{aa}K+W=0$ ，求联系数 K 。

④ 求改正数 $V=QA^TK$ ，平差值 $\hat{L}=L+V$

⑤ 检验平差计算的正确性。

注意事项：

- a) 条件方程应选择变量少、形式简单的列出；
- b) 条件方程线性化时，角度用系数 ρ 来统一单位。

弧度与 $^{\circ}$ 的转换

π 弧度= 180°

$$1\text{弧度} = \frac{180^{\circ}}{3.14} = 57.3^{\circ} = \rho^{\circ}$$

$$= 3437' = \rho'$$

$$= 206265'' = \rho''$$

例1：如图独立同精度测角网, 试按条件平差法求各个内角的平差值。

$$L_1 = 80^\circ 00' 06'', L_2 = 50^\circ 00' 00'', L_3 = 50^\circ 00' 00'', L_4 = 40^\circ 00' 00'', L_5 = 40^\circ 00' 00'', L_6 = 99^\circ 59' 57''.$$

①建立函数模型。

立列 r 个条件方程 $F(\hat{L})=0$,
线性化变换为 $AV+W=0$ 。

②建立随机模型。取 σ_0^2 ,
确定观测值协因数阵 Q 。

③建立法方程。 $N_{aa}K+W=0$,
求联系数 K 。

④求改正数 $V=QA^TK$, 平差
值 $\hat{L}=L+V$

⑤检验平差计算正确性。

解：① $n = 6, t = 4, r = 2$, 可
列出两个条件方程：

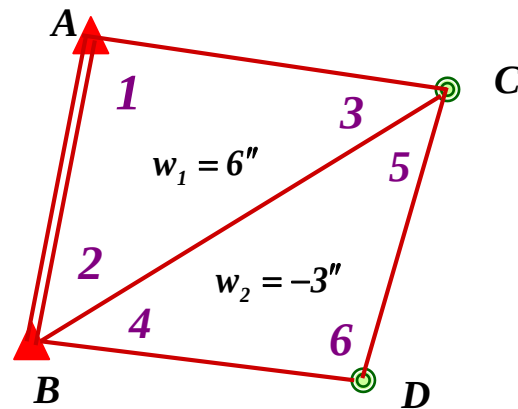
$$\begin{cases} \hat{L}_1 + \hat{L}_2 + \hat{L}_3 - 180^\circ = 0 \\ \hat{L}_4 + \hat{L}_5 + \hat{L}_6 - 180^\circ = 0 \end{cases}$$

以 $\hat{L}_i = L_i + v_i$ 和 L_i 的值带入方程

$$v_1 + v_2 + v_3 + \frac{L_1 + L_2 + L_3 - 180}{1} = 0$$

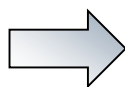
$$v_4 + v_5 + v_6 + \frac{L_4 + L_5 + L_6 - 180}{1} = 0$$

$$\begin{cases} v_1 + v_2 + v_3 + 6'' = 0 \\ v_4 + v_5 + v_6 - 3'' = 0 \end{cases}$$



$$\begin{matrix} AV + W = 0 \\ \begin{matrix} r1 & n1 & r1 \end{matrix} \end{matrix}$$

$$\begin{cases} v_1 + v_2 + v_3 + 6'' = 0 \\ v_4 + v_5 + v_6 - 3'' = 0 \end{cases}$$



$$A_{26} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} + \begin{pmatrix} 6'' \\ -3'' \end{pmatrix} = 0$$

$$W_{21} = \begin{pmatrix} +6'' \\ -3'' \end{pmatrix}$$

① **建立函数模型。**

立列 r 个条件方程 $F(\hat{L})=0$,
线性化变换为 $AV+W=0$ 。

② **建立随机模型。**取 σ_0^2 ,
确定观测值协因数阵 Q 。

③ **建立法方程。** $N_{aa}K+W=0$,
求联系数 K 。

④ **求改正数** $V=QA^TK$, 平差
值 $\hat{L}=L+V$

⑤ **检验平差计算正确性。**

②取 $\sigma_0^2=\sigma_\beta^2$, 因为观测值独立同精度, 则观测值的协因素阵 Q 、权阵 P 为

$$Q_{ii} = \sigma_i^2 / \sigma_0^2 = \sigma_\beta^2 / \sigma_\beta^2 = 1 \quad Q_{ij} = \sigma_{ij} / \sigma_0^2 = 0$$

$$Q_{66} = I_{66}$$

$$P_{66} = Q_{66}^{-1} = I_{66}$$

③建立法方程，求K。

①建立函数模型。

立列r个条件方程 $F(\hat{L})=0$,
线性化变换为 $AV+W=0$ 。

②建立随机模型。取 σ_0^2 ,
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求联系数 K 。

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值 $\hat{L}=L+V$

⑤检验平差计算正确性。

$$A_{26} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$Q_{66} = P_{66}^{-1} = I_{66}$$

$$N_{22}^{aa} = AQA^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} I \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\text{法方程: } \begin{cases} 3k_1 + 6'' = 0 \\ 3k_2 - 3'' = 0 \end{cases} \longrightarrow k_1, k_2$$

$$N_{22}^{aa^{-1}} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$K_{21} = -N_{aa}^{-1}W = -\frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} +6 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ +1 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

④求改正数 $V=QAK$, 平差值 $\hat{L}=L+V$

①建立函数模型。

立列 r 个条件方程 $F(\hat{L})=0$,
线性化变换为 $AV+W=0$ 。

②建立随机模型。取 σ_0^2 ,
确定观测值协因数阵 Q 。

③建立法方程。 $N_{aa}K+W=0$,
求联系数 K 。

④求改正数 $V=QA^TK$, 平差
值 $\hat{L}=L+V$

⑤检验平差计算正确性。

$$V_{61} = QA^TK = I \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ +1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \\ +1 \\ +1 \\ +1 \end{pmatrix} (")$$

$$Q = P_{66}^{-1} = I_{66}$$

$$A_{26} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$K_{21} = \begin{pmatrix} -2 \\ +1 \end{pmatrix}$$

$$\hat{L}_{61} = L + V = \begin{pmatrix} 80^\circ 00' 06'' \\ 50^\circ 00' 00'' \\ 50^\circ 00' 00'' \\ 40^\circ 00' 00'' \\ 40^\circ 00' 00'' \\ 99^\circ 59' 57'' \end{pmatrix} + \begin{pmatrix} -2'' \\ -2'' \\ -2'' \\ +1'' \\ +1'' \\ +1'' \end{pmatrix} = \begin{pmatrix} 80^\circ 00' 04'' \\ 49^\circ 59' 58'' \\ 49^\circ 59' 58'' \\ 40^\circ 00' 01'' \\ 40^\circ 00' 01'' \\ 99^\circ 59' 58'' \end{pmatrix}$$

分配了 w

⑤检验平差计算正确性。将 $\hat{L}_i$ 代入条件方程, 满足等式。求解无误

以上是本节课的内容。

同学们，下节课再见！

第四章 条件平差

§4-1 条件平差原理

§4-2 平差结果精度评价

§4-3 条件方程类型

§4-4 条件平差实例

§4-2 平差结果精度评价

测量平差的任务: 1) 求 $\mu = \tilde{L}$ 最优估值 $\hat{L}$;

2) 求 σ_0^2 的估计 $\hat{\sigma}_0^2$, 评价 L 、 $\hat{L}$ 、 $\varphi(\hat{L})$ 的精度。

下面从3方面进行

1、单位权方差 σ_0^2 的估计

2、平差基本变量的协因数阵 Q_{ZZ} , $D_{ZZ} = \hat{\sigma}_0^2 Q_{ZZ}$

3、平差值函数 $\varphi(\hat{L})$ 的方差

§4-2 平差结果精度评价

1、单位权方差 σ_0^2 的估计

由独立同精度的 Δ_i 估算 σ^2

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \Delta_i^2$$

由独立不同精度的 Δ_i 估算 σ_0^2

$$\hat{\sigma}_0^2 = \frac{1}{n} \sum_{i=1}^n p_i \Delta_i^2$$

可以证明（§8-1），单位权方差 σ_0^2 的无偏估计量为

$$\hat{\sigma}_0^2 = \frac{V^T P V}{r} \stackrel{\text{不相关观测}}{=} \frac{1}{r} [P V V] = \frac{1}{r} \sum_{i=1}^n p_i v_i^2$$

$$P_{\text{不相关}} = \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & \ddots \\ & & & p_n \end{pmatrix}$$

r —多余观测数； P —观测值权阵， V —改正数

观测值的方差估值为 $\hat{\sigma}_i^2 = \frac{\hat{\sigma}_0^2}{p_i}$

§4-2 平差结果精度评价

2、平差基本变量的协因数阵 Q_{ZZ}

$$Z = \begin{pmatrix} L_{n1} \\ W_{r1} \\ K_{r1} \\ V_{n1} \\ \hat{L}_{n1} \end{pmatrix} \quad \begin{aligned} L &= I L \\ W &= A L + A_0 \\ K &= -N_{aa}^{-1} W = -N_{aa}^{-1} A L - N_{aa}^{-1} A_0 \\ V &= Q A^T K = -Q A^T N_{aa}^{-1} W \\ \hat{L} &= L + V \end{aligned}$$

$$Z = \begin{pmatrix} L \\ W \\ K \\ V \\ \hat{L} \end{pmatrix} \begin{pmatrix} L \\ W \\ K \\ V \\ \hat{L} \end{pmatrix} = Z \quad Q_{ZZ} = \begin{pmatrix} Q_{LL} & Q_{LW} & Q_{LK} & Q_{LV} & Q_{L\hat{L}} \\ Q_{WL} & Q_{WW} & Q_{WK} & Q_{WV} & Q_{W\hat{L}} \\ Q_{KL} & Q_{KW} & Q_{KK} & Q_{KV} & Q_{K\hat{L}} \\ Q_{VL} & Q_{VW} & Q_{VK} & Q_{VV} & Q_{V\hat{L}} \\ Q_{\hat{L}L} & Q_{\hat{L}W} & Q_{\hat{L}K} & Q_{\hat{L}V} & Q_{\hat{L}\hat{L}} \end{pmatrix}$$

对称

$$Q_{LL} = Q$$

$$Q_{WW} = A Q A^T = N_{aa}$$

$$Q_{KK} = N_{aa}^{-1} Q_{WW} N_{aa}^{-1} = N_{aa}^{-1} N_{aa} N_{aa}^{-1} = N_{aa}^{-1}$$

$$Q_{VV} = Q A^T Q_{KK} A Q = Q A^T N_{aa}^{-1} A Q$$

$$Q_{LW} = Q A^T$$

$$Q_{LK} = -Q A^T N_{aa}^{-1}$$

$$Q_{LV} = -Q A^T N_{aa}^{-1} A Q$$

$$Q_{WK} = -A Q A^T N_{aa}^{-1} = -I$$

$$Q_{WV} = Q_{WW} N_{aa}^{-1} A Q = N_{aa} N_{aa}^{-1} A Q = A Q$$

$$Q_{KV} = N_{aa}^{-1} Q_{WW} N_{aa}^{-1} A Q = N_{aa}^{-1} A Q$$

$$Q_{L\hat{L}} = Q_{LL} + Q_{LV} = Q - Q A^T N_{aa}^{-1} A Q$$

$$Q_{W\hat{L}} = Q_{WL} + Q_{WV} = Q_{LW}^T + Q_{WV} = -A Q + A Q = 0$$

$$Q_{K\hat{L}} = Q_{KL} + Q_{KV} = -N_{aa}^{-1} A Q + N_{aa}^{-1} A Q = 0$$

$$Q_{V\hat{L}} = Q_{VL} + Q_{VV} = 0$$

$$Q_{\hat{L}\hat{L}} = Q_{LL} + Q_{LV} + Q_{VL} + Q_{VV} = Q - Q A^T N_{aa}^{-1} A Q$$

$$Z = \begin{pmatrix} L \\ W \\ K \\ V \\ \hat{L} \end{pmatrix} \rightarrow \begin{pmatrix} L \\ W \\ K \\ V \\ \hat{L} \end{pmatrix} = Z$$

$\hat{L}$ 与X的协因数通式

$$Q_{\hat{L}X} = Q_{LX} + Q_{VX}$$

$$Q_{X\hat{L}} = Q_{XL} + Q_{XV}$$

Z的方差阵

$$D_{ZZ} = \hat{\sigma}_0^2 Q_{ZZ}$$

条件平差基本变量的协因数表

	L	W	K	V	$\hat{L}$
L	Q	QA^T	$-QA^TN_{aa}^{-1}$	$-Q_{VV}$	$Q-Q_{VV}$
W	AQ	N_{aa}	$-I$	$-AQ$	0
K	$-N_{aa}^{-1}AQ$	$-I$	N_{aa}^{-1}	$N_{aa}^{-1}AQ$	0
V	$-Q_{VV}$	$-QA^T$	$QA^TN_{aa}^{-1}$	$QA^TN_{aa}^{-1}AQ$	0
$\hat{L}$	$Q-Q_{VV}$	0	0	0	$Q-Q_{VV}$

$\hat{L}$ 与W、K、V不相关

§4-2 平差结果精度评价

3、平差值函数的方差估计

平差值 $\hat{L}$ 的 m 个函数 $\varphi(\hat{L})$

$$\varphi = \varphi(\hat{L}) = \begin{pmatrix} \varphi_1(\hat{L}) & \varphi_2(\hat{L}) & \cdots & \varphi_m(\hat{L}) \end{pmatrix}^T$$

求全微分，得系数矩阵

$$d\varphi = \left(\frac{\partial \varphi}{\partial \hat{L}} \right)_0 d\hat{L} = f_{mn}^T d\hat{L}_{n1}$$

由协因数传播定理，得

$$\begin{aligned} Q_{\varphi\varphi} &= f_{mn}^T Q_{\hat{L}\hat{L}} f_{nm} \\ &= f^T Q f - f^T Q A^T N_{aa}^{-1} A Q f \end{aligned}$$

求各函数的方差估值

$$\hat{\sigma}_{\varphi_i}^2 = \hat{\sigma}_0^2 Q_{\varphi_i\varphi_i} \quad i = 1, 2, \dots, m$$

$$f_{mn}^T = \left(\frac{\partial \varphi}{\partial \hat{L}} \right)_0 = \begin{pmatrix} \frac{\partial \varphi_1}{\partial \hat{L}_1} & \frac{\partial \varphi_1}{\partial \hat{L}_2} & \cdots & \frac{\partial \varphi_1}{\partial \hat{L}_n} \\ \frac{\partial \varphi_2}{\partial \hat{L}_1} & \frac{\partial \varphi_2}{\partial \hat{L}_2} & \cdots & \frac{\partial \varphi_2}{\partial \hat{L}_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial \hat{L}_1} & \frac{\partial \varphi_m}{\partial \hat{L}_2} & \cdots & \frac{\partial \varphi_m}{\partial \hat{L}_n} \end{pmatrix}_0$$

$$Q_{\hat{L}\hat{L}} = Q - Q A^T N_{aa}^{-1} A Q$$

以上是本节课的内容。

同学们，下节课再见！

第四章 条件平差

§4-1 条件平差原理

§4-2 平差结果精度评价

§4-3 条件方程类型

§4-4 条件平差实例

§4-3 条件方程类型

条件方程是条件平差的函数模型，是平差解算过程的关键。

函数模型与观测类别、已知条件、观测情况、引入参数等多种因素有关。

- 1、水准网条件方程
 - 2、测角网条件方程
 - 3、测边网条件方程
 - 4、边角网条件方程
-

1、水准网条件方程

如图水准网已知点、未知点个数： $p_0=1$, $p=3$

观测值个数、必要观测、多余观测个数：

$$n=5, \quad t=p=3, \quad r=n-t=2$$

可列出 $r=2$ 个独立的条件方程：

$$\begin{cases} \hat{h}_1 + \hat{h}_4 + \hat{h}_5 = 0 \\ \hat{h}_2 + \hat{h}_3 - \hat{h}_4 = 0 \end{cases}$$

还有方程吗？

$$\hat{h}_1 + \hat{h}_2 + \hat{h}_3 + \hat{h}_5 = 0$$

令 $\hat{h}_i = h_i + v_i$ ，将观测值代入方程得

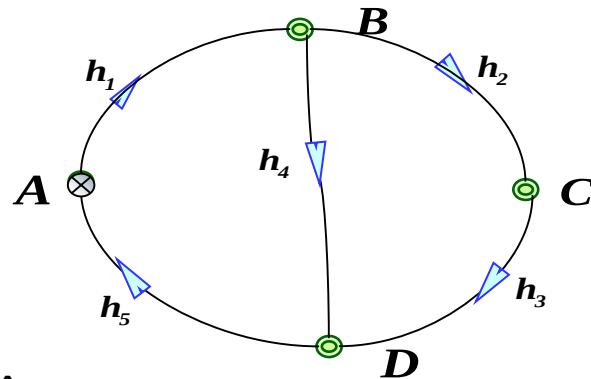
$$\begin{cases} v_1 + v_4 + v_5 + w_1 = 0 \\ v_2 + v_3 - v_4 + w_2 = 0 \end{cases}$$

$$w_1 = h_1 + h_4 + h_5$$

$$w_2 = h_2 + h_3 - h_4$$

$$A_{25} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 \end{pmatrix}$$

$$W_{21} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$



当已知点个数 $p_0=0$ 时
则需假设一个点为已知
点，如取其高程为
100.0m。

2、测角网条件方程

如图中**点多边形**测角网，已知点、未知点个数： $p_0=2$, $p=2$
观测值个数、必要观测、多余观测个数：

$$n=9, \quad t=2p=4, \quad r=n-t=5$$

应列出 $r=5$ 个**独立**条件方程，包括**三种条件方程**：

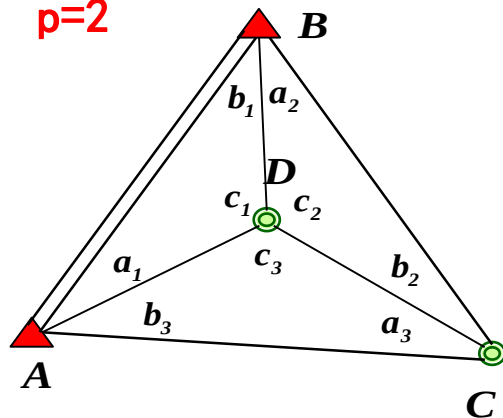
(1) **图形条件（内角和条件）3个**

$$\begin{cases} \hat{a}_1 + \hat{b}_1 + \hat{c}_1 - 180^\circ = 0 \\ \hat{a}_2 + \hat{b}_2 + \hat{c}_2 - 180^\circ = 0 \\ \hat{a}_3 + \hat{b}_3 + \hat{c}_3 - 180^\circ = 0 \end{cases}$$

令 $\hat{a}_i = a_i + v_{a_i}$ ，得**改正数**条件方程为：

$$\begin{cases} v_{a_1} + v_{b_1} + v_{c_1} + w_1 = 0 \\ v_{a_2} + v_{b_2} + v_{c_2} + w_2 = 0 \\ v_{a_3} + v_{b_3} + v_{c_3} + w_3 = 0 \end{cases}$$

$$w_i = a_i + b_i + c_i - 180^\circ \quad i = 1, 2, 3$$



(2) 圆周条件1个

$$\hat{c}_1 + \hat{c}_2 + \hat{c}_3 - 360^\circ = 0$$

改正数条件方程为：

$$v_{c_1} + v_{c_2} + v_{c_3} + w_4 = 0$$

$$w_4 = c_1 + c_2 + c_3 - 360^\circ$$

(3) 极条件（边的闭合关系）1个

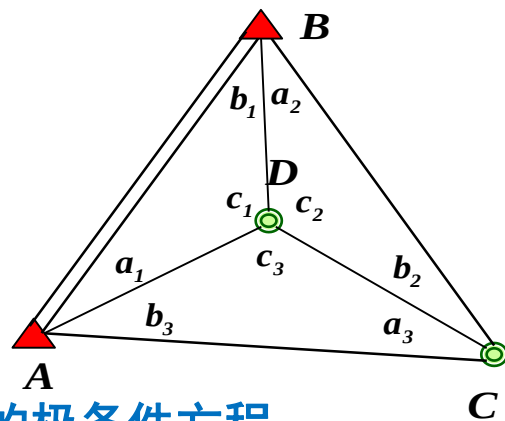
以D点为极，得到边的闭合关系，转化为角量关系的极条件方程

$$\frac{DB}{DA} \cdot \frac{DA}{DC} \cdot \frac{DC}{DB} = 1$$

$$\frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3} = 1$$

线性化转化

$$\cot a_1 v_{a_1} + \cot a_2 v_{a_2} + \cot a_3 v_{a_3} - \cot b_1 v_{b_1} - \cot b_2 v_{b_2} - \cot b_3 v_{b_3} + w_5 = 0$$



$$w_5 = \rho'' \left(1 - \frac{\sin b_1 \sin b_2 \sin b_3}{\sin a_1 \sin a_2 \sin a_3} \right)$$

正弦商条件方程（极条件、边长条件）的线性化

条件方程

$$\frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3} = k$$

$$\implies F(\hat{a}, \hat{b}) - k = 0$$

令 $F(\hat{a}, \hat{b}) = \frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3}$

$$F(a, b) = \frac{\sin a_1 \sin a_2 \sin a_3}{\sin b_1 \sin b_2 \sin b_3}$$

$$F(\hat{a}, \hat{b}) = \frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3}$$

$$= F(a, b) + \left(\frac{\partial F}{\partial \hat{a}}\right)_0 (\hat{a} - a) + \left(\frac{\partial F}{\partial \hat{b}}\right)_0 (\hat{b} - b)$$

$$= F(a, b) + \left(\frac{\partial F}{\partial \hat{a}}\right)_0 \mathbf{v}_a + \left(\frac{\partial F}{\partial \hat{b}}\right)_0 \mathbf{v}_b$$

$F(\hat{a}, \hat{b})$ 在 a, b 处线性化, $\hat{a}_i = a_i + v_{a_i}, \hat{b}_i = b_i + v_{b_i}$

$$i = 1, 2, 3$$

$$\left(\frac{\partial F}{\partial \hat{a}_1}\right)_0 = \left(\frac{\cos \hat{a}_1}{\sin \hat{a}_1} \frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3}\right)_0 \quad \left(\frac{\partial F}{\partial \hat{b}_1}\right)_0 = \left(-\frac{\cos \hat{b}_1}{\sin \hat{b}_1} \frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3}\right)_0$$

$$= \frac{\cos a_1}{\sin a_1} \frac{\sin a_1 \sin a_2 \sin a_3}{\sin b_1 \sin b_2 \sin b_3}$$

$$= \cot a_1 F(a, b)$$

$$\left(\frac{\partial F}{\partial \hat{a}_i}\right)_0 = \cot a_i F(a, b)$$

$$= -\frac{\cos b_1}{\sin b_1} \frac{\sin a_1 \sin a_2 \sin a_3}{\sin b_1 \sin b_2 \sin b_3}$$

$$= -\cot b_1 F(a, b)$$

$$\left(\frac{\partial F}{\partial \hat{b}_i}\right)_0 = -\cot b_i F(a, b) \quad i = 1, 2, 3$$

代入条件方程 $F(\hat{a}, \hat{b}) - k = 0$

$$F(a, b) + F(a, b) \cot a_1 \mathbf{v}_{a_1} + F(a, b) \cot a_2 \mathbf{v}_{a_2} + F(a, b) \cot a_3 \mathbf{v}_{a_3} - F(a, b) \cot b_1 \mathbf{v}_{b_1} + F(a, b) \cot b_2 \mathbf{v}_{b_2} + F(a, b) \cot b_3 \mathbf{v}_{b_3} - k = 0$$

$$\cot a_1 \mathbf{v}_{a_1} + \cot a_2 \mathbf{v}_{a_2} + \cot a_3 \mathbf{v}_{a_3} - \cot b_1 \mathbf{v}_{b_1} - \cot b_2 \mathbf{v}_{b_2} - \cot b_3 \mathbf{v}_{b_3} + w = 0$$

$$w = \rho'' \left[1 - \frac{k}{F(a, b)} \right]$$

$\mathbf{v}$ 单位取角秒

正弦商方程（极条件、边长条件）的线性化-示例

将下列平差值方程化为改正数方程

$$(1) \frac{\sin(\hat{X}_1 - \hat{L}_1)}{\sin \hat{L}_2} \cdot \frac{\sin \hat{L}_3}{\sin \hat{L}_4} \cdot \frac{\sin \hat{L}_5}{\sin \hat{L}_6} = 1$$

$$\text{令 } \hat{L}_i = L_i + v_i, \quad \hat{X}_1 = X_1^0 + \hat{x}_1 \quad i = 1, 2, \dots, 6$$

$$\begin{aligned} & \cot(X_1^0 - L_1)(\hat{x}_1 - v_1) - \cot L_2 v_2 + \cot L_3 v_3 \\ & - \cot L_4 v_4 + \cot L_5 v_5 - \cot L_6 v_6 + w_1 = 0 \end{aligned}$$

$$w_1 = \rho'' \left[1 - \frac{\sin L_2 \sin L_4 \sin L_6}{\sin(X_1^0 - L_1) \sin L_3 \sin L_5} \right]$$

$$(2) \frac{S_{AB}}{\sin \hat{L}_2} = \frac{S_0}{\sin \hat{L}_4} \quad \Rightarrow \quad \frac{\sin \hat{L}_4}{\sin \hat{L}_2} = \frac{S_0}{S_{AB}}$$

$$\text{令 } \hat{L}_i = L_i + v_i$$

$$-\cot L_2 v_2 + \cot L_4 v_4 + w_2 = 0$$

$$w_2 = \rho'' \left[1 - \frac{S_0 \sin L_2}{S_{AB} \sin L_4} \right]$$

$$\frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3} = k$$



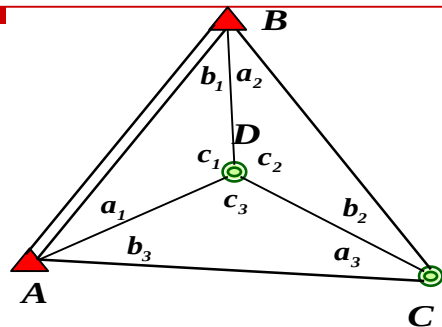
$$\begin{aligned} & \cot a_1 v_{a_1} + \cot a_2 v_{a_2} + \cot a_3 v_{a_3} \\ & - \cot b_1 v_{b_1} - \cot b_2 v_{b_2} - \cot b_3 v_{b_3} + w = 0 \end{aligned}$$

$$w = \rho'' \left[1 - \frac{k}{F(a, b)} \right]$$

极条件产生的三种情形

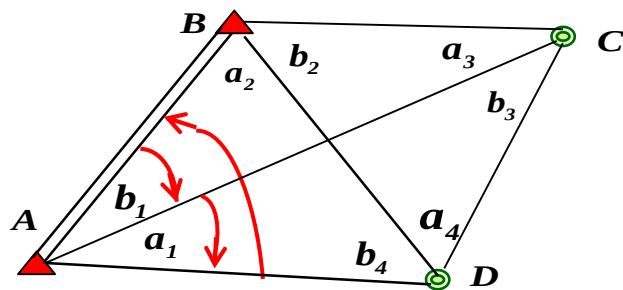
①中点多边形
$$\frac{\sin \hat{a}_1 \sin \hat{a}_2 \sin \hat{a}_3}{\sin \hat{b}_1 \sin \hat{b}_2 \sin \hat{b}_3} = 1$$

以D为极:
$$\frac{DB}{DA} \cdot \frac{DA}{DC} \cdot \frac{DC}{DB} = 1$$

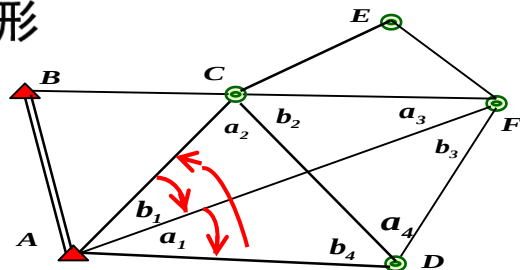


②大地四边形
$$\frac{\sin \hat{a}_3 \sin(\hat{a}_4 + \hat{b}_4) \sin \hat{a}_2}{\sin(\hat{a}_2 + \hat{b}_2) \sin \hat{b}_3 \sin \hat{b}_4} = 1$$

以A为极:
$$\frac{AB}{AC} \cdot \frac{AC}{AD} \cdot \frac{AD}{AB} = 1$$



③其他混合图形



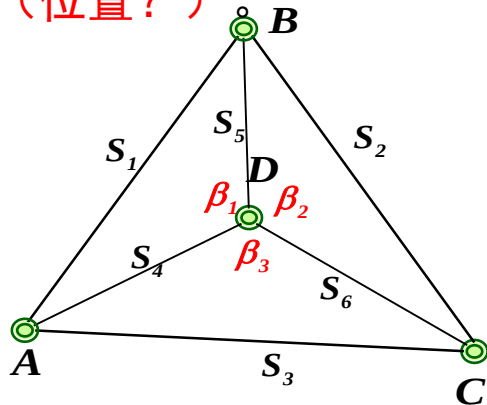
3、测边网条件方程

如图中点多边形测边网，需确定三角网的形状、大小（位置？）
边长观测值 $s_1, s_2, \dots, s_6$. 边长平差值 $\hat{s}_1, \hat{s}_2, \dots, \hat{s}_6$.

$$n=6, t=3+2=5, r=n-t=1$$

可列1个边长平差值条件方程 $F(\hat{s})=0$ ，但不易列出。

故先列角度 $\hat{\beta}$ 的条件方程，然后转化为边长 $\hat{s}$ 的条件方程。



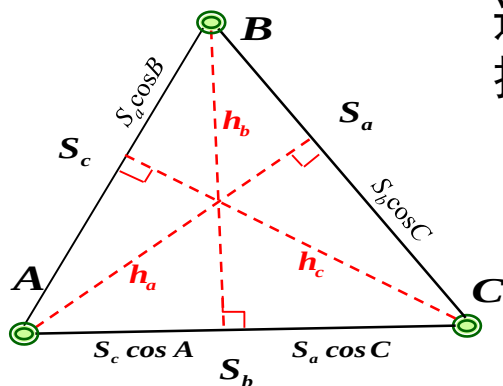
1) $\hat{\beta}$ 的条件方程 $\hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3 - 360^\circ = 0$

$$\begin{array}{c} \hat{\beta}_i = \beta_i + v_{\beta_i} \\ \xrightarrow{i=1, 2 \text{ L } 6} v_{\beta_1} + v_{\beta_2} + v_{\beta_3} + w = 0 \\ w = \beta_1 + \beta_2 + \beta_3 - 360^\circ \end{array}$$

对于测边网，如需确定位置：则需1个已知点坐标+已知方向1个。

3、测边网条件方程

2) 单三角形边角（改正数）关系



$$S_b - S_c \cos A = S_a \cos C$$

$$S_c - S_b \cos A = S_a \cos B$$

$$S_b S_c \sin A = S_b h_b = S_a h_a$$

边长观测值 S_a, S_b, S_c 边长平差值 $\hat{S}_a, \hat{S}_b, \hat{S}_c$.
按余弦定理得边角等式，经微分变换，有

$$\hat{S}_a^2 = \hat{S}_b^2 + \hat{S}_c^2 - 2\hat{S}_b\hat{S}_c \cos \hat{A}$$

$$\begin{aligned} 2S_a d\hat{S}_a &= 2(S_b - S_c \cos A) d\hat{S}_b + 2(S_c - S_b \cos A) d\hat{S}_c + 2S_b S_c \sin A d\hat{A} \\ &= 2S_a \cos C d\hat{S}_b + 2S_a \cos B d\hat{S}_c + 2S_a h_a d\hat{A} \end{aligned}$$

$$d\hat{A} = \frac{1}{h_a} (d\hat{S}_a - \cos C d\hat{S}_b - \cos B d\hat{S}_c)$$

用改正数替换微分，考虑角度改正数取''

$$v_A'' = \frac{\rho''}{h_a} (v_{S_a} - \cos C v_{S_b} - \cos B v_{S_c})$$

-同侧角边量

3、测边网条件方程

3) 以边长改正数表示角度改正数条件方程

按 v_A 表达式的规律，可得 v_{β_i} 的表达式

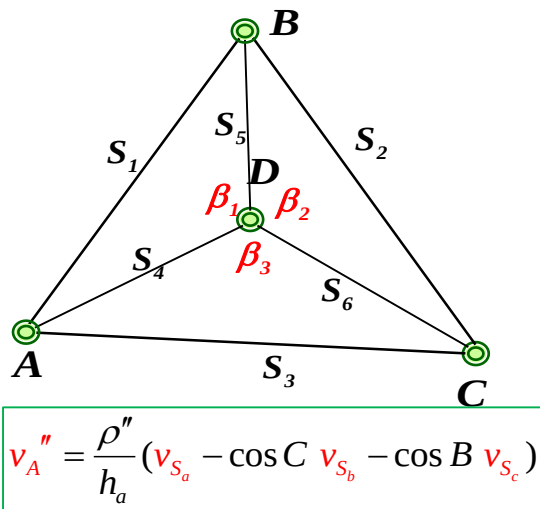
$$v_{\beta_1}'' = \frac{\rho''}{h_1} (v_{S_1} - \cos \angle DAB v_{S_4} - \cos \angle DBA v_{S_5})$$

$$v_{\beta_2}'' = \frac{\rho''}{h_2} (v_{S_2} - \cos \angle DBC v_{S_5} - \cos \angle DCB v_{S_6})$$

$$v_{\beta_3}'' = \frac{\rho''}{h_3} (v_{S_3} - \cos \angle DAC v_{S_4} - \cos \angle DCA v_{S_6})$$

将 v_{β_i} 代入 $v_{\beta_1} + v_{\beta_2} + v_{\beta_3} + w = 0$ 得关于 v_{S_i} 的条件方程：

$$\begin{aligned} & \frac{\rho''}{h_1} v_{S_1} + \frac{\rho''}{h_2} v_{S_2} + \frac{\rho''}{h_3} v_{S_3} - \rho'' \left(\frac{1}{h_1} \cos \angle DAB + \frac{1}{h_3} \cos \angle DAC \right) v_{S_4} \\ & - \rho'' \left(\frac{1}{h_1} \cos \angle DBA + \frac{1}{h_2} \cos \angle DBC \right) v_{S_5} - \rho'' \left(\frac{1}{h_1} \cos \angle DCB + \frac{1}{h_3} \cos \angle DCA \right) v_{S_6} + w = 0 \end{aligned}$$



-同侧角边量

式中角度近似值，可以由边长观测值按余弦定理或其它公式求得。

4、边角网条件方程

如图边角网，需确定三角网的形状、大小。

观测值 $S_1, S_2, S_3, \beta_1, \beta_2, \beta_3$; 平差值 $\hat{S}_1, \hat{S}_2, \hat{S}_3, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$

$$n=6, t=3, r=n-t=3$$

可列3个平差值条件方程及转化为改正数方程:

内角和条件方程1个: $\hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3 - 180^\circ = 0$

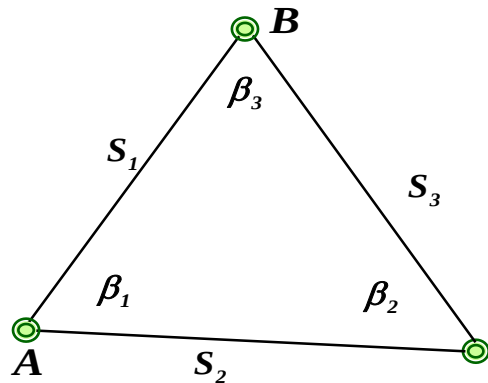
$$\text{令 } \hat{\beta}_i = \beta_i + v_{\beta_i} \quad \hat{S}_i = S_i + v_{S_i} \quad i=1, 2, 3$$

$$v_{\beta_1} + v_{\beta_2} + v_{\beta_3} + w = 0 \quad w = \beta_1 + \beta_2 + \beta_3 - 180^\circ$$

正弦定理条件方程1个: $\frac{\hat{S}_1}{\sin \hat{\beta}_2} - \frac{\hat{S}_2}{\sin \hat{\beta}_3} = 0 \Rightarrow \hat{S}_1 \sin \hat{\beta}_3 - \hat{S}_2 \sin \hat{\beta}_2 = 0$

$$\begin{aligned} & -S_2 \cos \beta_2 v_{\beta_2} + S_1 \cos \beta_3 v_{\beta_3} + \rho'' \sin \beta_3 v_{S_1} \\ & - \rho'' \sin \beta_2 v_{S_2} + \rho'' (S_1 \sin \beta_3 - S_2 \sin \beta_2) = 0 \end{aligned}$$

余弦定理条件方程1个: $\hat{S}_1^2 = \hat{S}_2^2 + \hat{S}_3^2 - 2\hat{S}_2\hat{S}_3 \cos \hat{\beta}_2$



4、边角网条件方程

余弦定理条件： $\hat{S}_1^2 = \hat{S}_2^2 + \hat{S}_3^2 - 2\hat{S}_2\hat{S}_3 \cos \hat{\beta}_2$

$$S_1 d\hat{S}_1 = (S_2 - S_3 \cos \beta_2) d\hat{S}_2 + (S_3 - S_2 \cos \beta_2) d\hat{S}_3 + S_2 S_3 \sin \beta_2 d\hat{\beta}_2$$

$$v_{\beta_2}'' = \frac{\rho''}{S_2 S_3 \sin \beta_2} [S_1 v_{S_1} - (S_2 - S_3 \cos \beta_2) v_{S_2} - (S_3 - S_2 \cos \beta_2) v_{S_3}]$$

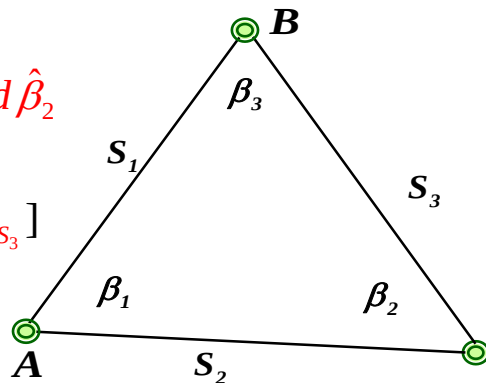
由几何关系知

$$S_2 S_3 \sin \beta_2 = S_1 h_1$$

$$\frac{S_1}{S_1 h_1} = \frac{1}{S_1} (\cot \beta_1 + \cot \beta_3)$$

$$\frac{S_2 - S_3 \cos \beta_2}{S_1 h_1} = \frac{\cos \beta_1}{h_1} = \frac{1}{S_2} \cot \beta_1$$

$$\frac{S_3 - S_2 \cos \beta_2}{S_1 h_1} = \frac{\cos \beta_3}{h_1} = \frac{1}{S_3} \cot \beta_3$$



$$\Rightarrow v_{\beta_2}'' = \frac{\rho''}{S_1} (\cot \beta_1 + \cot \beta_3) v_{S_1} - \frac{\rho''}{S_2} \cot \beta_1 v_{S_2} - \frac{\rho''}{S_3} \cot \beta_2 v_{S_3}$$

以上是本节课的内容。

同学们，下节课再见！

第四章 条件平差

§4-1 条件平差原理

§4-2 平差结果精度评价

§4-3 条件方程类型

§4-4 条件平差实例

实例1：水准网条件平差

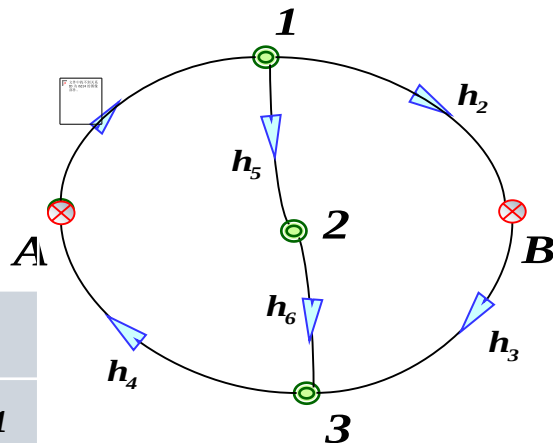
如图水准网已知点A、B，起算数据、观测数据于表。试按条件平差法1) 求出观测值的平差值及其协因数阵；2) 2点高程的平差值及其中误差。

高程观测值及线路长度表

测段	1	2	3	4	5	6
$h_i(m)$	14.301	17.074	-20.220	-11.159	12.821	-15.961
$S_i(km)$	2.2	3.1	2.8	3.7	2.7	2.1

$$H_A = 150.372m$$

$$H_B = 181.739m$$



解：① 未知点3个： $p_0=2$, $p=3$

观测值个数、必要观测、多余观测个数：

$$n=6, \quad t=p=3, \quad r=n-t=3$$

①建立函数模型。

立列 r 个条件方程 $F(\hat{L})=0$,
线性化变换为 $AV+W=0$ 。

②建立随机模型。取 σ_0^2 ,
确定观测值协因数阵 Q 。

③建立法方程。 $N_{aa}K+W=0$,
求联系数 K 。

④求改正数 $V=QA^TK$, 平差
值 $\hat{L}=L+V$

⑤检验平差计算正确性。

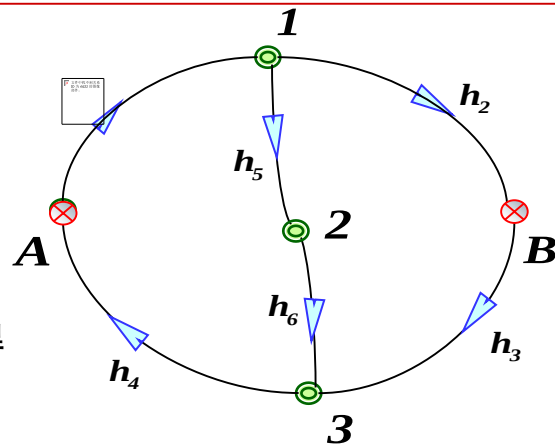
$$\begin{cases} \hat{h}_1 + \hat{h}_4 + \hat{h}_5 + \hat{h}_6 = 0 \\ \hat{h}_2 + \hat{h}_3 - \hat{h}_5 - \hat{h}_6 = 0 \\ \hat{h}_1 + \hat{h}_2 + H_A - H_B = 0 \end{cases}$$

令 $\hat{h}_i = h_i + v_i$, 代入观测值得

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} + \begin{pmatrix} 2 \\ -6 \\ 8 \end{pmatrix} = 0$$

$$A_{36} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_{31} = \begin{pmatrix} 3 \\ -6 \\ 8 \end{pmatrix} (mm)$$



②**建立随机模型**。取 $C=5\text{km}$ ，即 5km 水准路线高差观测值为单

位权观测值

$$\sigma_0^2 = (1/C) \sigma_{km}^2,$$

高程观测值的**权和协因数**

$$p_i = \frac{5}{S_i}$$

$$Q_{ii} = \frac{S_i}{5} \quad Q_{ij} = 0$$

高差观测值的协因数阵为

$$Q = Q_{hh} = \begin{pmatrix} 0.44 & & & & & \\ & 0.62 & & & & \\ & & 0.56 & & & \\ & & & 0.74 & & \\ & & & & 0.54 & \\ & & & & & 0.42 \end{pmatrix}$$

测段	1	2	3	4	5	6
$h_i(\text{m})$	14.301	17.074	-20.220	-11.159	12.821	-15.961
$S_i(\text{km})$	2.2	3.1	2.8	3.7	2.7	2.1
$Q_{ii}=S_i/5$	0.44	0.62	0.56	0.74	0.54	0.42
$p_i=5/S_i$	2.273	1.613	1.786	1.351	1.852	2.381

③建立法方程求K

$$\begin{aligned}
 A_{36} &= \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} & Q_{66} = Q_{hh} &= \begin{pmatrix} 0.44 & & & & & \\ & 0.62 & & & & \\ & & 0.56 & & & \\ & & & 0.74 & & \\ & & & & 0.54 & \\ & & & & & 0.42 \end{pmatrix} & W_{31} &= \begin{pmatrix} 3 \\ -6 \\ 8 \end{pmatrix} (mm) \\
 N_{aa} &= AQA^T = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.44 & & & & & \\ & 0.62 & & & & \\ & & 0.56 & & & \\ & & & 0.74 & & \\ & & & & 0.54 & \\ & & & & & 0.42 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0.44 & 0 & 0 & 0.74 & 0.54 & 0.42 \\ 0 & 0.62 & 0.56 & 0 & -0.54 & -0.42 \\ 0.44 & 0.62 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2.14 & -0.96 & 0.44 \\ -0.96 & 2.14 & 0.62 \\ 0.44 & 0.62 & 1.06 \end{pmatrix}
 \end{aligned}$$

$$N_{rr}^{aa} \mathbf{K} + \mathbf{W}_{r1} = 0$$

①建立函数模型。

立列 r 个条件方程 $F(\hat{L})=0$,
线性化变换为 $\mathbf{A}\mathbf{V}+\mathbf{W}=0$ 。

②建立随机模型。取 σ_0^2 ,
确定观测值协因数阵 $\mathbf{Q}$ 。

③建立法方程。 $N_{aa}\mathbf{K}+\mathbf{W}=0$,
求联系数 $\mathbf{K}$ 。

④求改正数 $\mathbf{V}=\mathbf{Q}\mathbf{A}^T\mathbf{K}$, 平差
值 $\hat{L}=\mathbf{L}+\mathbf{V}$

⑤检验平差计算正确性。

$$\begin{pmatrix} 2.14 & -0.96 & 0.44 \\ -0.96 & 2.14 & 0.62 \\ 0.44 & 0.62 & 1.06 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} + \begin{pmatrix} 2 \\ -6 \\ 8 \end{pmatrix} = 0$$

$$\mathbf{K} = [7.69 \quad 11.27 \quad 17.33]^T \text{ (mm)}$$

④求改正数 $\mathbf{V}=\mathbf{Q}\mathbf{A}^T\mathbf{K}$ 、观测值的平差值

$$\mathbf{V} = [-4.2 \quad -3.7 \quad 6.3 \quad 5.7 \quad -1.9 \quad -1.5]^T \text{ (mm)}$$

$$\hat{h} = [14.2968 \quad 17.0703 \quad -20.2137 \quad -11.1533 \quad 12.8191 \quad -15.9625]^T \text{ (m)}$$

⑤检验。平差值 $\hat{h}$ 代入条件方程，满足等式要求。

单位权中误差的估值

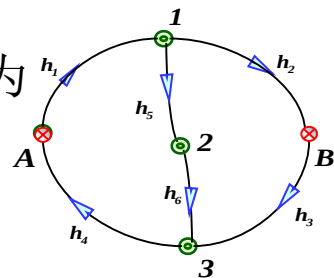
$$\hat{\sigma}_0 = \sqrt{\frac{V^T P V}{r}} = \sqrt{\frac{\sum_{i=1}^n p_i v_i^2}{r}} = 8mm$$

观测值平差值的协因数阵为

$$Q_{\hat{h}\hat{h}} = Q - Q A^T N_{aa}^{-1} A Q$$

$$= \begin{pmatrix} 0.2142 & -0.2142 & 0.0534 & -0.0534 & -0.0905 & -0.0704 \\ & 0.2142 & -0.0534 & 0.0534 & 0.0905 & 0.0704 \\ & & 0.2526 & -0.2526 & 0.1121 & 0.0872 \\ & & & 0.2526 & -0.1121 & -0.0872 \\ & & & & 0.3502 & -0.1476 \\ & & & & & 0.3052 \end{pmatrix}$$

第2点高程的平差值为



$$\hat{H}_2 = H_A + \hat{h}_1 + \hat{h}_5$$

$$= (1 \ 0 \ 0 \ 0 \ 1 \ 0) (\hat{h}_1 \ \hat{h}_2 \ \hat{h}_3 \ \hat{h}_4 \ \hat{h}_5 \ \hat{h}_6)^T$$

$$f^T = [1 \ 0 \ 0 \ 0 \ 1 \ 0]$$

$$\frac{1}{p_{\hat{H}_2}} = Q_{\hat{H}_2 \hat{H}_2} = f^T Q_{\hat{h}\hat{h}} f = 0.3835$$

第2点高程平差值的中误差为

$$\hat{\sigma}_{\hat{H}_2} = \hat{\sigma}_0 \sqrt{Q_{\hat{H}_2 \hat{H}_2}} = 4.9mm$$

小结与作业

- 1、平差解题关键在于按照原理和步骤，认真解算。
- 2、反映解题思路的必要过程需要简要表述，运算的过程则可简略表述。

作业：

习题四**P70**：第**1、2、3、5、9、11**题

以上是本节课的内容。

同学们，下节课再见！

第五章 附有参数的条件平差

§5-1 附有参数的条件平差原理

§5-2 平差结果精度评价

§5-2 附有参数的条件平差实例

引入参数情况→函数模型形式→平差方法类型→平差结果不变

引入参数 $\hat{X}$ 个数 u	函数模型个数 c	函数模型形式	函数模型名称	平差方法
$u=0$	$c=r$	$F_{r1}(\hat{L}_{n1}) = 0$	条件方程	条件平差(4)
$u < t$ 独立	$c=r+u$	$F_{c1}(\hat{L}_{n1}, \hat{X}_{u1}) = 0$	附有参数的条件方程	附有参数的条件平差(5)
$u=t$ 独立	$c=r+t=n$	$\hat{L}_{n1} = F_{n1}(\hat{X}_{t1})$	误差方程	间接平差(6)
$u > t$ 其中 $s=u-t$ 不独立	$c=r+u$ $=n+s$	$\hat{L}_{n1} = F_{n1}(\hat{X}_{u1})$ $\Phi_{s1}(\hat{X}_{u1}) = 0$	误差方程+限制条件方程	附有限制条件的间接平差(7)

平差的任务—由观测值 L 求平差值 $\hat{L}$ （观测量真值 $\tilde{L}$ 的估值）

§5-1 附有参数的条件平差原理

1、附有参数的条件平差原理(由L求 $\hat{L}$, $\hat{X}$?)

一个平差问题, r个多余观测就产生r个函数模型;

现引入u ($u < t$) 个独立参数 $\hat{X}$, 又增加u个函数模型, 使得函数模型个数变为 $c=r+u$ 个。函数模型的形式为:

$$\begin{cases} \text{非线性: } F(\hat{L}, \hat{X}) = 0 \\ \text{线性: } A \hat{L} + B \hat{X} + A_0 = 0 \end{cases} \xrightarrow[\hat{X} = X^0 + \hat{x}]{\hat{L} = L + V} \begin{matrix} \text{cn} & \text{n1} & \text{cu} & \text{u1} & \text{c1} \\ A V + B \hat{x} + W = 0 \end{matrix}$$

改正数条件方程

$$\begin{aligned} F(\hat{L}, \hat{X}) &\approx F(L, X^0) + \left(\frac{\partial F}{\partial \hat{L}}\right)_0 (\hat{L} - L) + \left(\frac{\partial F}{\partial \hat{X}}\right)_0 (\hat{X} - X^0) \\ &= A V + B \hat{x} + W \end{aligned}$$

$$W_{c1} = F(L, X^0) = AL + BX^0 + A_0 \quad A = \left(\frac{\partial F}{\partial \hat{L}}\right)_0 \quad B = \left(\frac{\partial F}{\partial \hat{X}}\right)_0$$

1、平差原理(由W求V, $\hat{x}$?)

(几何关系)

$$\left. \begin{array}{l} \text{函数模型: } \mathbf{A} \mathbf{V} + \mathbf{B} \hat{\mathbf{x}} + \mathbf{W} = \mathbf{0} \\ \text{平差原则: } \mathbf{V}^T \mathbf{P} \mathbf{V} = \min \end{array} \right\} \Rightarrow \mathbf{V}, \hat{\mathbf{x}} = ?$$

这是一条件极值问题:

在 $\mathbf{A}\mathbf{V} + \mathbf{B}\mathbf{x} + \mathbf{W} = 0$ 的条件下,
求 $\mathbf{V}^T \mathbf{P} \mathbf{V} = \min$ 的极值点 $\mathbf{V}, \hat{\mathbf{x}}$ 。

$$\text{随机模型: } \mathbf{D} = \sigma_0^2 \mathbf{P}^{-1} = \sigma_0^2 \mathbf{Q}$$

$$\Phi(\mathbf{V}) = \frac{\mathbf{V}^T \mathbf{P} \mathbf{V}}{11} - 2 \frac{\mathbf{K}^T (\mathbf{A}\mathbf{V} + \mathbf{B}\hat{\mathbf{x}} + \mathbf{W})}{1c} \quad 0$$

为此构建一目标函数:

$$\text{待定的联系数矩阵 } \mathbf{K}_{c1} = (\mathbf{k}_1, \mathbf{k}_2 \text{ L } \mathbf{k}_c)^T$$

现求合适的 $\mathbf{V}, \hat{\mathbf{x}}$, 使 $\Phi(\mathbf{V}) = \min$ 。必要条件是:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial \mathbf{V}_{n1}} \Phi(\mathbf{V})_{11} = 2\mathbf{V}^T \mathbf{P} - 2\mathbf{K}^T \mathbf{A} = 0 \\ \frac{\partial}{\partial \hat{\mathbf{x}}_{u1}} \Phi(\mathbf{V})_{11} = -2\mathbf{K}^T \mathbf{B} = 0 \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{V}_{n1} = \mathbf{Q} \mathbf{A}^T \mathbf{K}_{c1} \\ \mathbf{B}_{uc}^T \mathbf{K}_{c1} = 0 \end{array} \right. \quad (\text{极值关系})$$

1、平差原理(由W求V, $\hat{x}$) $\rightarrow \hat{L}, \hat{X}$

$$\begin{aligned}
 & \left(\text{几何关系} \right) \left\{ \begin{aligned} & \frac{\partial}{\partial V_{n1}} \Phi(V) = 2V^T P - 2K^T A = 0 \\ & \frac{\partial}{\partial \hat{x}_{u1}} \Phi(V) = -2K^T B = 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} & \begin{matrix} V \\ n1 \end{matrix} = QA^T \begin{matrix} K \\ c1 \end{matrix} \\ & \begin{matrix} B^T \\ uc \end{matrix} \begin{matrix} K \\ c1 \end{matrix} = 0 \end{aligned} \right\} \left\{ \begin{aligned} & N_{cc} \begin{matrix} K \\ c1 \end{matrix} + \begin{matrix} B \\ cu \end{matrix} \begin{matrix} \hat{x} \\ u1 \end{matrix} + \begin{matrix} W \\ c1 \end{matrix} = 0 \\ & \begin{matrix} B^T \\ uc \end{matrix} \begin{matrix} K \\ c1 \end{matrix} = 0 \end{aligned} \right. \quad \begin{matrix} K = -N_{aa}^{-1}(B\hat{x} + W) \\ \nearrow \end{matrix}
 \end{aligned}$$

(极值关系) 法方程

$$B^T N_{aa}^{-1} B \hat{x} + B^T N_{aa}^{-1} W = 0 \quad N_{bb} = B_{uc}^T N_{aa}^{-1} B_{cu} \quad N_{cc} = AQA^T \quad R(N_{bb}) = u$$

$$\hat{x} = -N_{bb}^{-1} B^T N_{aa}^{-1} W$$

$$\hat{X} = X^0 + \hat{x}$$

$$K = -N_{aa}^{-1}(B\hat{x} + W)$$

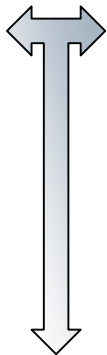
$$V = QA^T K$$

$$\hat{L} = L + V$$

1、平差原理（矩阵式-代数式）

条件方程 $F(\hat{L}, \hat{X}) = 0$

$$\begin{cases} F_1(\hat{L}_1, \hat{L}_2 \cdots \hat{L}_n, \hat{X}_1, \cdots \hat{X}_u) = 0 \\ F_2(\hat{L}_1, \hat{L}_2 \cdots \hat{L}_n, \hat{X}_1, \cdots \hat{X}_u) = 0 \\ \cdots \\ F_c(\hat{L}_1, \hat{L}_2 \cdots \hat{L}_n, \hat{X}_1, \cdots \hat{X}_u) = 0 \end{cases}$$



$$\underset{cn \ n1}{A} \hat{L} + \underset{cu \ u1}{B} \hat{X} + \underset{c1}{A_0} = 0$$

$$\begin{cases} a_{11}\hat{L}_1 + a_{12}\hat{L}_2 + \cdots + a_{1n}\hat{L}_n + b_{11}\hat{X}_1 + \cdots + b_{1u}\hat{X}_u + A_{10} = 0 \\ a_{21}\hat{L}_1 + a_{22}\hat{L}_2 + \cdots + a_{2n}\hat{L}_n + b_{21}\hat{X}_1 + \cdots + b_{2u}\hat{X}_u + A_{20} = 0 \\ \cdots \quad \cdots \quad \cdots \\ a_{c1}\hat{L}_1 + a_{c2}\hat{L}_2 + \cdots + a_{cn}\hat{L}_n + b_{c1}\hat{X}_1 + \cdots + b_{cu}\hat{X}_u + A_{c0} = 0 \end{cases}$$

$$\underset{cn \ n1}{A} V + \underset{cu \ u1}{B} \hat{x} + \underset{c1}{W} = 0$$

$$\begin{cases} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n + b_{11}\hat{x}_1 + \cdots + b_{1u}\hat{x}_u + w_1 = 0 \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n + b_{21}\hat{x}_1 + \cdots + b_{2u}\hat{x}_u + w_2 = 0 \\ \cdots \quad \cdots \quad \cdots \\ a_{c1}v_1 + a_{c2}v_2 + \cdots + a_{cn}v_n + b_{c1}\hat{x}_1 + \cdots + b_{cu}\hat{x}_u + w_c = 0 \end{cases}$$

$$W = F(L, X^0) = \begin{pmatrix} F_1(L, X^0) \\ F_2(L, X^0) \\ \vdots \\ F_c(L, X^0) \end{pmatrix}$$

$$\underset{cn}{A} = \left(\frac{\partial F_{c1}}{\partial \hat{L}_{n1}} \right)_0 = \begin{pmatrix} \frac{\partial F_1}{\partial \hat{L}_1} & \frac{\partial F_1}{\partial \hat{L}_2} & \cdots & \frac{\partial F_1}{\partial \hat{L}_n} \\ \frac{\partial F_2}{\partial \hat{L}_1} & \frac{\partial F_2}{\partial \hat{L}_2} & \cdots & \frac{\partial F_2}{\partial \hat{L}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_c}{\partial \hat{L}_1} & \frac{\partial F_c}{\partial \hat{L}_2} & \cdots & \frac{\partial F_c}{\partial \hat{L}_n} \end{pmatrix}_0 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{c1} & a_{c2} & \cdots & a_{cn} \end{pmatrix}$$

$$\underset{cu}{B} = \left(\frac{\partial F_{c1}}{\partial \hat{X}_{u1}} \right)_0 = \begin{pmatrix} \frac{\partial F_1}{\partial \hat{X}_1} & \frac{\partial F_1}{\partial \hat{X}_2} & \cdots & \frac{\partial F_1}{\partial \hat{X}_u} \\ \frac{\partial F_2}{\partial \hat{X}_1} & \frac{\partial F_2}{\partial \hat{X}_2} & \cdots & \frac{\partial F_2}{\partial \hat{X}_u} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_c}{\partial \hat{X}_1} & \frac{\partial F_c}{\partial \hat{X}_2} & \cdots & \frac{\partial F_c}{\partial \hat{X}_u} \end{pmatrix}_0 = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1u} \\ b_{21} & b_{22} & \cdots & b_{2u} \\ \cdots & \cdots & \cdots & \cdots \\ b_{c1} & b_{c2} & \cdots & b_{cu} \end{pmatrix}$$

2、附有参数条件平差计算步骤

①建立函数模型。

立列c个附有参数的条件方程 $F_{c1}(\hat{L}_{n1}, \hat{X}_{u1}) = 0$
，线性化变换为：
$$A_{cn\ n1} V + B_{cu\ u1} \hat{x} + W_{c1} = 0$$

②建立随机模型。取 σ_0^2 ，确定观测值协因数阵Q。

③建立法方程。
$$\begin{cases} N_{aa} K + B_{cu\ u1} \hat{x} + W_{c1} = 0 \\ B_{uc}^T K = 0 \end{cases}$$

求 $\hat{x}, K$ 。

④求改正数 $V = QA^TK$ ，
平差值 $\hat{L} = L + V$, $\hat{X} = X^0 + \hat{x}$

⑤检验平差计算正确性。

注意事项：

- a) 引入参数要合理， $u < t$ ，独立
- b) 附有参数条件方程应选择变量少、形式简单的列出；
- c) 条件方程线性化时，角度用系数 ρ 来统一单位。

弧度与 $^{\circ}$ 的转换

π 弧度 $=180^{\circ}$

$$\begin{aligned} 1\text{弧度} &= \frac{180^{\circ}}{3.14} = 57.3^{\circ} = \rho^{\circ} \\ &= 3437' = \rho' \\ &= 206265'' = \rho'' \end{aligned}$$

以上是本节课的内容。

同学们，下节课再见！

第五章 附有参数的条件平差

§5-1 附有参数的条件平差原理

§5-2 附有参数条件平差精度评价

§5-2 附有参数的条件平差实例

§5-2 平差结果精度评价

测量平差的任务: 1) 求最优估值 $\hat{L}, \hat{X}$;

2) 求 σ_0^2 的估计 $\hat{\sigma}_0^2$, 评价 L 、 $\hat{L}$ 、 $\hat{X}$ 、 $\varphi(\hat{L}, \hat{X})$ 的精度。

下面从3方面进行

1、单位权方差 σ_0^2 的估计

2、平差基本变量的协因数阵 Q_{ZZ} , $D_{ZZ} = \hat{\sigma}_0^2 Q_{ZZ}$

3、平差值函数 $\varphi(\hat{L}, \hat{X})$ 的中误差

§5-2 平差结果精度评价

1、单位权方差 σ_0^2 的估计

可以证明，单位权方差 σ_0^2 的无偏估计量为

$$\hat{\sigma}_0^2 = \frac{V^T P V}{r} \stackrel{\text{不相关观测}}{=} \frac{1}{r} [P V V] = \frac{1}{r} \sum_{i=1}^n p_i v_i^2$$
$$P_{\text{不相关}} = \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & \ddots \\ & & & p_n \end{pmatrix}$$

r —多余观测数； P —观测值权阵， V —改正数

观测值的方差估值为

$$\hat{\sigma}_i^2 = \frac{\hat{\sigma}_0^2}{p_i}$$

§5-2 平差结果精度评价

2、平差基本变量的协因数阵 Q_{ZZ}

$$Z = \begin{pmatrix} L_{n1} \\ W_{r1} \\ \hat{X}_{u1} \\ K_{r1} \\ V_{n1} \\ \hat{L}_{n1} \end{pmatrix}$$

$$L = I \text{ } L$$

$$W = F(L, X^0) = A \text{ } L + B X^0 + A_0$$

$$\hat{X} = X^0 + \hat{x} = X^0 - N_{bb}^{-1} B^T N_{aa}^{-1} W$$

$$K = -N_{aa}^{-1} W - N_{aa}^{-1} B \hat{X}$$

$$= (-N_{aa}^{-1} + N_{aa}^{-1} B N_{bb}^{-1} B^T N_{aa}^{-1}) W$$

$$V = Q A^T K$$

$$\hat{L} = L + V$$

$$Z = \begin{pmatrix} L \\ W \\ \hat{X} \\ K \\ V \\ \hat{L} \end{pmatrix} = \begin{pmatrix} L \\ W \\ \hat{X} \\ K \\ V \\ \hat{L} \end{pmatrix} = Z$$

$$Q_{ZZ} = \begin{pmatrix} Q_{LL} & Q_{LW} & Q_{L\hat{X}} & Q_{LK} & Q_{LV} & Q_{L\hat{L}} \\ Q_{WL} & Q_{WW} & Q_{W\hat{X}} & Q_{WK} & Q_{WV} & Q_{W\hat{L}} \\ Q_{\hat{X}L} & Q_{\hat{X}W} & Q_{\hat{X}\hat{X}} & Q_{\hat{X}K} & Q_{\hat{X}V} & Q_{\hat{X}\hat{L}} \\ Q_{KL} & Q_{KW} & Q_{K\hat{X}} & Q_{KK} & Q_{KV} & Q_{K\hat{L}} \\ Q_{VL} & Q_{VW} & Q_{V\hat{X}} & Q_{VK} & Q_{VV} & Q_{V\hat{L}} \\ Q_{\hat{L}L} & Q_{\hat{L}W} & Q_{\hat{L}\hat{X}} & Q_{\hat{L}K} & Q_{\hat{L}V} & Q_{\hat{L}\hat{L}} \end{pmatrix}$$

§5-2 平差结果精度评价

2、平差基本变量的协因数阵 Q_{ZZ}

$$Z = \begin{pmatrix} L \\ n1 \\ W \\ r1 \\ \hat{X} \\ u1 \\ K \\ r1 \\ V \\ n1 \\ \hat{L} \\ n1 \end{pmatrix}$$

$$L = I \quad L$$

$$W = F(L, X^0) = AL + BX^0 + A_0$$

$$\hat{X} = X^0 + \hat{x} = X^0 - N_{bb}^{-1} B^T N_{aa}^{-1} W$$

$$K = -N_{aa}^{-1} W - N_{aa}^{-1} B \hat{x}$$

$$= (-N_{aa}^{-1} + N_{aa}^{-1} B N_{bb}^{-1} B^T N_{aa}^{-1}) W$$

$$V = QA^T K$$

$$\hat{L} = L + V$$

$$Q_{ZZ} = \begin{pmatrix} Q_{LL} & Q_{LW} & Q_{L\hat{X}} & Q_{LK} & Q_{LV} & Q_{L\hat{L}} \\ Q_{WL} & Q_{WW} & Q_{W\hat{X}} & Q_{WK} & Q_{WV} & Q_{W\hat{L}} \\ Q_{\hat{X}L} & Q_{\hat{X}W} & Q_{\hat{X}\hat{X}} & Q_{\hat{X}K} & Q_{\hat{X}V} & Q_{\hat{X}\hat{L}} \\ Q_{KL} & Q_{KW} & Q_{K\hat{X}} & Q_{KK} & Q_{KV} & Q_{K\hat{L}} \\ Q_{VL} & Q_{VW} & Q_{V\hat{X}} & Q_{VK} & Q_{VV} & Q_{V\hat{L}} \\ Q_{\hat{L}L} & Q_{\hat{L}W} & Q_{\hat{L}\hat{X}} & Q_{\hat{L}K} & Q_{\hat{L}V} & Q_{\hat{L}\hat{L}} \end{pmatrix}$$

$$Q_{LL} = Q$$

$$Q_{WW} = AQA^T = N_{aa}$$

$$Q_{\hat{X}\hat{X}} = -N_{bb}^{-1} B^T N_{aa}^{-1} Q_{WW} (-N_{aa}^{-1} B N_{bb}^{-1}) = N_{bb}^{-1}$$

$$Q_{KK} = (-N_{aa}^{-1} + N_{aa}^{-1} B N_{bb}^{-1} B^T N_{aa}^{-1}) Q_{WW}$$

$$(-N_{aa}^{-1} + N_{aa}^{-1} B N_{bb}^{-1} B^T N_{aa}^{-1})^T$$

$$= N_{aa}^{-1} - N_{aa}^{-1} B Q_{\hat{X}\hat{X}} B^T N_{aa}^{-1}$$

$$Q_{VV} = QA^T Q_{KK} AQ$$

$$Q_{\hat{L}\hat{L}} = Q_{LL} + Q_{LV} + Q_{VL} + Q_{VV} = Q - Q_{VV}$$

$$Q_{LW} = QA^T$$

$$Q_{L\hat{X}} = -QA^T N_{aa}^{-1} BQ_{\hat{X}\hat{X}}$$

$$Q_{LK} = -QA^T Q_{KK}$$

$$Q_{LV} = -Q_{VV}$$

$$Q_{L\hat{L}} = Q - Q_{VV}$$

$$Q_{W\hat{X}} = -BQ_{\hat{X}\hat{X}}$$

$$Q_{WK} = -N_{aa} Q_{\hat{X}\hat{X}}$$

$$Q_{WV} = -N_{aa} Q_{KK} A Q$$

$$Q_{W\hat{L}} = BQ_{\hat{X}\hat{X}} B^T N_{aa}^{-1} A Q$$

$\hat{L}$ 与 Y 的协因数通式

$$Q_{\hat{L}Y} = Q_{LY} + Q_{VY}$$

$$Q_{Y\hat{L}} = Q_{YL} + Q_{YV}$$

$$Q_{\hat{X}\hat{X}} = Q_{\hat{x}\hat{x}}$$

$$Q_{\hat{X}K} = -N_{bb}^{-1}B^T N_{aa}^{-1}Q_{WW}(-N_{aa}^{-1} + N_{aa}^{-1}BN_{bb}^{-1}B^T N_{aa}^{-1})^T = O$$

$$Q_{\hat{X}V} = O$$

$$Q_{\hat{X}\hat{L}} = Q_{\hat{X}L} + Q_{\hat{X}V} = -N_{bb}^{-1}B^T N_{aa}^{-1}AQ$$

$$Q_{KV} = Q_{KK}AQ$$

$$Q_{K\hat{L}} = Q_{KL} + Q_{KV} = O$$

$$Q_{V\hat{L}} = Q_{VL} + Q_{VV} = O$$

$\hat{L}$ 与 Y 的协因数通式

$$Q_{\hat{L}Y} = Q_{LY} + Q_{VY}$$

$$Q_{Y\hat{L}} = Q_{YL} + Q_{YV}$$

$$Q_{\hat{X}\hat{X}} = Q_{\hat{x}\hat{x}}$$

Z的方差阵

$$D_{ZZ} = \hat{\sigma}_0^2 Q_{ZZ}$$

附有参数的条件平差基本变量的协因数表

	L	W	X	K	V	$\hat{L}$
L	Q	QA^T	$-QA^TN_{aa}^{-1}BQ_{XX}$	$-QA^TQ_{KK}$	$-Q_{VV}$	$Q-Q_{VV}$
W	AQ	N_{aa}	$-BQ_{XX}$	$-N_{aa}Q_{KK}$	$-N_{aa}Q_{KK}$ AQ	$BQ_{XX}B^TN_{aa}^{-1}AQ$
X	$Q_{XX}B^TN_{aa}^{-1}AQ$	$-Q_{XX}B^T$	N_{bb}^{-1}	O	O	$-N_{bb}^{-1}B^TN_{aa}^{-1}AQ$
K	$-Q_{KK}AQ$	$-Q_{KK}N_{aa}$	O	$N_{aa}^{-1}-N_{aa}^{-1}$ $BQ_{XX}B^TN_{aa}^{-1}$	$Q_{KK}AQ$	O
V	$-Q_{VV}$	$-QA^TQ_{KK}N_{aa}$	O	QA^TQ_{KK}	$QA^TN_{KK}^{-1}AQ$	O
$\hat{L}$	$Q-Q_{VV}$	$QA^TN_{aa}^{-1}BQ_{XX}B^T$	$-QA^TN_{aa}^{-1}$ BN_{bb}^{-1}	O	O	$Q-Q_{VV}$

$\hat{L}$ 与 K 、 V 不相关, $\hat{X}$ 与 K 、 V 不相关

§5-2 平差结果精度评价

3、平差值函数的方差估计

平差值 $\hat{L}$, $\hat{X}$ 的 m 个函数 $\varphi(\hat{L}, \hat{X})$

对 $\varphi(\hat{L}, \hat{X})$ 求全微分, 得系数矩阵

$$\varphi = \varphi(\hat{L}, \hat{X}) = \begin{pmatrix} \varphi_1(\hat{L}, \hat{X}) \\ \varphi_2(\hat{L}, \hat{X}) \\ \dots \\ \varphi_m(\hat{L}, \hat{X}) \end{pmatrix}$$

$$d\varphi = \left(\frac{\partial \varphi}{\partial \hat{L}} \right)_0 d\hat{L} + \left(\frac{\partial \varphi}{\partial \hat{X}} \right)_0 d\hat{X} = f_{mn}^T d\hat{L}_{n1} + f_{xu}^T d\hat{X}_{u1} = \begin{pmatrix} f^T & f_x^T \end{pmatrix} \begin{pmatrix} d\hat{L} \\ d\hat{X} \end{pmatrix}$$

由协因数传播定理, 得

$$\begin{aligned} Q_{\varphi\varphi} &= \begin{pmatrix} f^T & f_x^T \end{pmatrix} \begin{pmatrix} Q_{\hat{L}\hat{L}} & Q_{\hat{L}\hat{X}} \\ Q_{\hat{X}\hat{L}} & Q_{\hat{X}\hat{X}} \end{pmatrix} \begin{pmatrix} f \\ f_x \end{pmatrix} \\ &= f^T Q_{\hat{L}\hat{L}} f + f^T Q_{\hat{L}\hat{X}} f_x + f_x^T Q_{\hat{X}\hat{L}} f + f_x^T Q_{\hat{X}\hat{X}} f_x \end{aligned}$$

$$f_{mn}^T = \left(\frac{\partial \varphi}{\partial \hat{L}} \right)_0$$

$$f_{xu}^T = \left(\frac{\partial \varphi}{\partial \hat{X}} \right)_0$$

求各函数的方差估值

$$\hat{\sigma}_{\varphi_i}^2 = \hat{\sigma}_0^2 Q_{\varphi_i \varphi_i} \quad i = 1, 2, \dots, m$$

以上是本节课的内容。

同学们，下节课再见！

第五章 附有参数的条件平差

§5-1 附有参数的条件平差原理

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§5-2 附有参数的条件平差实例

例1：如图测角网，A、B为已知点，A(100.0, 100.0)，B(100.0, 199.6208)，已知AD边长 $S_0=76.605\text{m}$ 。观测值 $L=(70^\circ00'06'' \ 55^\circ00'00'' \ 55^\circ00'00'' \ 40^\circ00'00'' \ 40^\circ00'00'' \ 99^\circ59'57'')^T$

①建立函数模型。

附参条件方程 $F(\hat{L}, \hat{X}) = 0$

线性化变换为：
$$\begin{matrix} \text{cn} & \text{nn} & \text{cn} & \text{nn} & \text{cu} & \text{u1} & \text{cu} & \text{u1} & \text{c1} \\ \mathbf{A} & \mathbf{V} + & \mathbf{B} & \hat{\mathbf{x}} + & \mathbf{W} & = & 0 \end{matrix}$$

②建立随机模型。取 σ_0^2 ，确定观测值协因数阵 $\mathbf{Q}$ 。

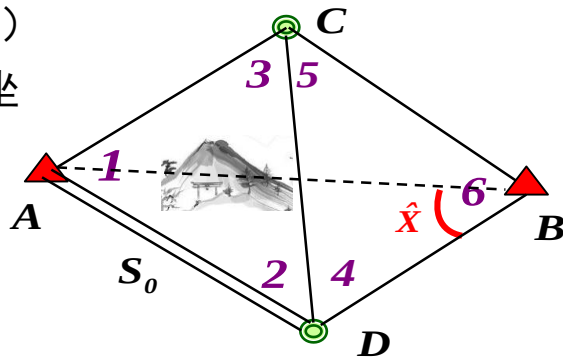
③建立法方程
$$\begin{cases} N_{cc} \mathbf{K} + \mathbf{B} \hat{\mathbf{x}} + \mathbf{W} = 0 \\ \mathbf{B}^T \mathbf{K} = 0 \end{cases}$$

④求改正数 $\mathbf{V} = \mathbf{Q} \mathbf{A}^T \mathbf{K}$ ，

平差值 $\hat{L} = L + \mathbf{V}$ ， $\hat{X} = X^0 + \hat{x}$

⑤检验平差计算正确性。

试按附有参数的条件平差法，1) 求各内角的平差值。2) 求D点坐标平差值及其中误差。



解：1) 求平差值 $\hat{L}$

① $n = 6, t = 4 - 1 = 3, r = 3$

由于AB不通视带来测角不完整，条件方程难于立列。不便采用条件平差。

为此，采用附有参数的条件平差。

①建立函数模型。

附参条件方程 $F(\hat{L}, \hat{X}) = 0$

线性化变换为: $\underset{cn \ n1}{A}V + \underset{cu \ u1}{B}\hat{x} + \underset{c1}{W} = 0$

②建立随机模型。取 σ_0^2 , 确定观测值协因数阵 Q 。

③建立法方程 $\begin{cases} N_{cc} \underset{c1}{K} + \underset{cu \ u1}{B} \hat{x} + \underset{c1}{W} = 0 \\ \underset{uc}{B}^T \underset{c1}{K} = 0 \end{cases}$

④求改正数 $V = QA^TK$,
平差值 $\hat{L} = L + V$, $\hat{X} = X^0 + \hat{x}$

⑤检验平差计算正确性。

引入1个参数, $u=1 < t$, $\hat{X} = \angle ABD$ $X^0 = 49^\circ 59' 58.6''$

函数模型个数 $c=r+u=4$, 附有参数的条件方程为

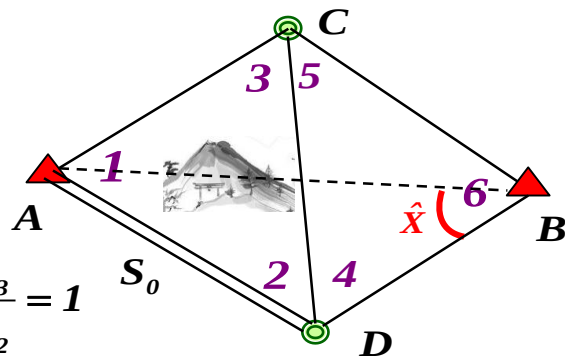
$$\hat{L}_1 + \hat{L}_2 + \hat{L}_3 - 180^\circ = 0$$

$$\hat{L}_4 + \hat{L}_5 + \hat{L}_6 - 180^\circ = 0$$

$$\frac{S_0}{\sin \hat{X}} = \frac{S_{AB}}{\sin(\hat{L}_2 + \hat{L}_4)}$$

$$\frac{\sin(\hat{L}_6 - \hat{X})}{\sin(\hat{L}_3 + \hat{L}_5)} \cdot \frac{\sin(\hat{L}_2 + \hat{L}_4)}{\sin \hat{X}} \cdot \frac{\sin \hat{L}_3}{\sin \hat{L}_2} = 1$$

$$\frac{AC}{AB} \cdot \frac{AB}{AD} \cdot \frac{AD}{AC} = 1$$



线性化转化为改正数方程

令 $\hat{L}_i = L_i + v_i, \hat{X} = X^0 + \hat{x}$

$i = 1, 2, \dots, 6$

$$\hat{L}_1 + \hat{L}_2 + \hat{L}_3 - 180^\circ = 0$$

$$v_1 + v_2 + v_3 + w_1 = 0$$

$$w_1 = L_1 + L_2 + L_3 - 180^\circ$$

$$\hat{L}_4 + \hat{L}_5 + \hat{L}_6 - 180^\circ = 0$$

$$v_4 + v_5 + v_6 + w_2 = 0$$

$$w_2 = L_4 + L_5 + L_6 - 180^\circ$$

$$\frac{\sin(\hat{L}_2 + \hat{L}_4)}{\sin \hat{X}} = \frac{S_{AB}}{S_0}$$

$$\cot(L_2 + L_4)v_2 + \cot(L_2 + L_4)v_4 - \cot X^0 \hat{x} + w_3 = 0$$

$$w_3 = \rho'' \left[1 - \frac{S_{AB}}{S_0} \frac{\sin X^0}{\sin(L_2 + L_4)} \right]$$

$$\frac{\sin(\hat{L}_6 - \hat{X})}{\sin(\hat{L}_3 + \hat{L}_5)} \cdot \frac{\sin(\hat{L}_2 + \hat{L}_4)}{\sin \hat{X}} \cdot \frac{\sin \hat{L}_3}{\sin \hat{L}_2} = 1$$

$$\begin{aligned} & [\cot(L_2 + L_4) - \cot L_2]v_2 + [\cot L_3 - \cot(L_3 + L_5)]v_3 + \cot(L_2 + L_4)v_4 \\ & - \cot(L_3 + L_5)v_5 + \cot(L_6 - X^0)v_6 - [\cot X^0 + \cot(L_6 - X^0)]\hat{x} + w_4 = 0 \end{aligned}$$

$$w_4 = \rho'' \left[1 - \frac{\sin(L_3 + L_5) \sin X^0 \sin L_2}{\sin(L_6 - X^0) \sin(L_2 + L_4) \sin L_3} \right]$$

$$\underset{cn \ n1}{\mathbf{A}} \mathbf{V} + \underset{cu \ u1}{\mathbf{B}} \hat{\mathbf{x}} + \underset{c1}{\mathbf{W}} = \mathbf{0}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -4.2416e-5 & 0 & -4.216e-5 & 0 & 0 \\ 0 & -0.7877 & 0.7877 & -0.0875 & 0.0875 & 0.8391 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -4.0682e-4 \\ 1.6782 \end{pmatrix} \hat{\mathbf{x}} + \begin{pmatrix} 6 \\ -3 \\ 0 \\ -5.04 \end{pmatrix} = \mathbf{0}$$

$$\underset{46}{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -4.2416e-5 & 0 & -4.216e-5 & 0 & 0 \\ 0 & -0.7877 & 0.7877 & -0.0875 & 0.0875 & 0.8391 \end{pmatrix} \quad \underset{41}{\mathbf{B}} = \begin{pmatrix} 0 \\ 0 \\ -4.0682e-4 \\ 1.6782 \end{pmatrix} \quad \underset{41}{\mathbf{W}} = \begin{pmatrix} 6 \\ -3 \\ 0 \\ -5.04 \end{pmatrix}$$

①建立函数模型。

附参条件方程 $F_{c1}(\hat{L}, \hat{X}) = 0$

线性化变换为: $A_{cn} V + B_{cu} \hat{x} + W_{c1} = 0$

②建立随机模型。取 σ_0^2 ，确定观测值协因数阵 Q 。

③建立法方程 $\begin{cases} N_{aa} K + B_{cu} \hat{x} + W_{c1} = 0 \\ B_{uc}^T K = 0 \end{cases}$

④求改正数 $V = QA^T K$,

平差值 $\hat{L} = L + V, \hat{X} = X^0 + \hat{x}$

⑤检验平差计算正确性。

②取 $\sigma_0^2 = \sigma_\beta^2$ ，考虑观测值独立，则观测值的协因数阵 $Q = I$ 。

③建立法方程

$$N_{aa} = AQA^T = \begin{pmatrix} 3 & 0 & -4.2416e-5 & 0 \\ 0 & 3 & -4.2416e-5 & 0.8391 \\ -4.258e-5 & -4.258e-5 & 3.6261e-9 & 3.727e-5 \\ 0 & 0.8391 & 3.7122e-5 & 1.9603 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & -4.2416e-5 & 0 \\ 0 & 3 & -4.2416e-5 & 0.8391 \\ -4.258e-5 & -4.258e-5 & 3.6261e-9 & 3.727e-5 \\ 0 & 0.8391 & 3.7122e-5 & 1.9603 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -4.0682e-4 \\ 1.6782 \end{pmatrix} \hat{x} + \begin{pmatrix} 6 \\ -3 \\ 0 \\ -5.04 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 0 & -4.0682e-4 & 1.6782 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = 0 \quad N_{aa}^{-1} = \begin{pmatrix} 0.5316 & 0.3096 & 1.4023e4 & -0.3981 \\ & 0.8631 & 2.1897e4 & -0.7837 \\ & & 9.9179e8 & -2.8154e4 \\ & & & 1.3787 \end{pmatrix}$$

$$\hat{x} = 0.13'' \quad K = (4.861 \times 10^{-4} \quad 6.804 \times 10^{-4} \quad -4.9136 \quad 0.0012)^T$$

④求改正数和平差值

$$\mathbf{V} = \mathbf{Q}\mathbf{A}^T \mathbf{K} = (1.8 \ -0.8 \ 5.1 \ -2.1 \ -2.1 \ 1.1)^T (")$$

$$\hat{\mathbf{L}} = \mathbf{L} + \mathbf{V} = (70^\circ 00' 07.8'' \ 54^\circ 59' 59.2'' \ 55^\circ 00' 5.1'' \ 39^\circ 59' 57.9'' \ 39^\circ 59' 57.9'' \ 99^\circ 59' 58.1'')^T$$

$$\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}} = 49^\circ 59' 58.7''$$

2) 求D点坐标平差值及其中误差

单位权中误差的估值为

$$\hat{\sigma}_0 = \sqrt{\frac{\mathbf{V}^T \mathbf{P} \mathbf{V}}{r}} = \sqrt{\frac{1}{r} \sum_{i=1}^n v_i^2} = 6.5''$$

参数 $\hat{\mathbf{X}}$ 、 $\hat{\mathbf{L}}$ 的协因数为

$$\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}} = \mathbf{N}_{bb}^{-1} = (\mathbf{B}^T \mathbf{N}_{aa}^{-1} \mathbf{B})^{-1} = 0.05 \quad \hat{\sigma}_{\hat{\mathbf{X}}} = \hat{\sigma}_0 \sqrt{\mathbf{Q}_{\hat{\mathbf{X}}\hat{\mathbf{X}}}} = 1.6''$$

$\hat{L}$ 的协因数和中误差为

$$Q_{\hat{L}\hat{L}} = Q - Q_{VV} = Q - QA^T Q_{KK} A Q \quad Q_{KK} = N_{aa}^{-1} - N_{aa}^{-1} B N_{bb}^{-1} B^T N_{aa}^{-1}$$

$$Q_{\hat{L}\hat{L}} = \begin{pmatrix} 0.6642 & -0.3621 & -0.3021 & -0.0107 & -0.0107 & 0.0215 \\ & 0.3361 & 0.0259 & -0.1234 & -0.6234 & 0.2468 \\ & & 0.2762 & 0.1341 & 0.1341 & 0.2683 \\ & & & 0.6206 & -0.3794 & -0.2412 \\ & & & & 0.6206 & -0.2412 \\ & & & & & 0.4824 \end{pmatrix}$$

$$\hat{\sigma}_{\hat{L}_1} = \hat{\sigma}_0 \sqrt{Q_{\hat{L}_1 \hat{L}_1}} = 5.0''$$

$$\hat{\sigma}_{\hat{L}_2} = \hat{\sigma}_0 \sqrt{Q_{\hat{L}_2 \hat{L}_2}} = 3.6''$$

$$\hat{\sigma}_{\hat{L}_3} = \hat{\sigma}_0 \sqrt{Q_{\hat{L}_3 \hat{L}_3}} = 3.3''$$

$$\hat{\sigma}_{\hat{L}_4} = \hat{\sigma}_0 \sqrt{Q_{\hat{L}_4 \hat{L}_4}} = 5.0''$$

$$\hat{\sigma}_{\hat{L}_5} = \hat{\sigma}_0 \sqrt{Q_{\hat{L}_5 \hat{L}_5}} = 5.0''$$

$$\hat{\sigma}_{\hat{L}_6} = \hat{\sigma}_0 \sqrt{Q_{\hat{L}_6 \hat{L}_6}} = 4.3''$$

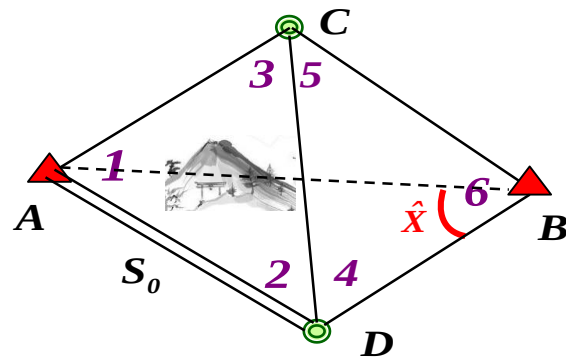
D点坐标的平差值的计算式φ

$$\begin{cases} X_D = X_A + S_0 \cos \alpha \\ Y_D = Y_A + S_0 \sin \alpha \end{cases}$$

$$\hat{\alpha} = \hat{\alpha}_{AD} = \alpha_{AB} - (180^\circ - \hat{X} - \hat{L}_2 - \hat{L}_4)$$

$$\begin{cases} dX_D = -S_0 \sin \alpha \frac{d\hat{L}_2}{\rho''} - S_0 \sin \alpha \frac{d\hat{L}_4}{\rho''} - S_0 \sin \alpha \frac{d\hat{X}}{\rho''} \\ dY_D = S_0 \cos \alpha \frac{d\hat{L}_2}{\rho''} + S_0 \cos \alpha \frac{d\hat{L}_4}{\rho''} + S_0 \sin \alpha \frac{d\hat{X}}{\rho''} \end{cases}$$

$$= \begin{pmatrix} 0 & -\frac{S_0 \sin \alpha}{\rho''} & 0 & -\frac{S_0 \sin \alpha}{\rho''} & 0 & 0 \\ 0 & \frac{S_0 \cos \alpha}{\rho''} & 0 & \frac{S_0 \cos \alpha}{\rho''} & 0 & 0 \end{pmatrix} \begin{pmatrix} d\hat{L}_1 \\ d\hat{L}_2 \\ \vdots \\ d\hat{L}_6 \end{pmatrix} + \begin{pmatrix} -\frac{S_0 \sin \alpha}{\rho''} \\ \frac{S_0 \sin \alpha}{\rho''} \end{pmatrix} d\hat{X} = f^T d\hat{L} + f_x^T d\hat{X}$$



D点坐标平差值的协因数阵

$$\mathbf{Q}_{DD} = \begin{pmatrix} f^T & f_x^T \end{pmatrix} \begin{pmatrix} \mathbf{Q}_{\hat{L}\hat{L}} & \mathbf{Q}_{\hat{L}\hat{X}} \\ \mathbf{Q}_{\hat{X}\hat{L}} & \mathbf{Q}_{\hat{X}\hat{X}} \end{pmatrix} \begin{pmatrix} f \\ f_x \end{pmatrix} = f^T \mathbf{Q}_{\hat{L}\hat{L}} f^T + f^T \mathbf{Q}_{\hat{L}\hat{X}} f_x^T + f_x^T \mathbf{Q}_{\hat{X}\hat{L}} f^T + f_x^T \mathbf{Q}_{\hat{X}\hat{X}} f_x^T$$

$$\mathbf{Q}_{DD} = \begin{pmatrix} 11.139 & -7.7997 \\ -7.7997 & 5.465 \end{pmatrix} \times 10^{-8}$$

$$\hat{\sigma}_{\hat{X}_D} = \hat{\sigma}_0 \sqrt{\mathbf{Q}_{\hat{X}_D \hat{X}_D}} = 4.1e-7 (dm)$$

$$\hat{\sigma}_{\hat{Y}_D} = \hat{\sigma}_0 \sqrt{\mathbf{Q}_{\hat{Y}_D \hat{Y}_D}} = 5.8e-7 (dm)$$

小结与作业

- 1、附有参数的条件平差，关键在独立参数的合理引入。
- 2、解题应反映解题思路的必要过程。

作业：

习题五P79：第2、5、7、8题

以上是本节课的内容。

同学们，下节课再见！
