
误差理论与测量平差基础

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平差方法、函数模型与引入参数个数的关系

引入参数 $\hat{X}$ 个数 u	函数模型			平差方法 类型
	个数	形式	名称	
$u=0$	$c=r$	$F_{r1}(\hat{L}_{n1}) = 0$	条件方程	条件平差(4)
$u < t$ 独立	$c=r+u$	$F_{c1}(\hat{L}_{n1}, \hat{X}_{u1}) = 0$	附有参数的条件方程	附有参数的条件平差(5)
$u=t$ 独立	$c=r+t=n$	$\hat{L}_{n1} = F_{n1}(\hat{X}_{t1})$	误差方程	间接平差(6)
$u > t$ 其中 $s=u-t$ 不独立	$c=r+u$ $=n+s$	$\hat{L}_{n1} = F_{n1}(\hat{X}_{u1})$ $\Phi_{s1}(\hat{X}_{u1}) = 0$	误差方程+限制 条件方程	附有限制条件的间接平差(7)

第六章 间接平差

§6-1 间接平差原理

§6-2 间接平差结果精度评价

§6-3 误差方程类型

§6-4 间接平差实例

§6-1 间接平差原理

讨论函数模型 ($\hat{L}, \hat{X}$ 的几何关系)

如图平差问题：观测值 L_1, L_2, L_3 ，求 L 的平差值 $\hat{L}_1, \hat{L}_2, \hat{L}_3$

1) 条件平差：不引入参数 $u=0$ ，函数模型个数 $c=r+u=1$

$$\hat{L}_1 + \hat{L}_2 + \hat{L}_3 - 180^\circ = 0 \quad F_{r1}(\hat{L}_{n1}) = 0$$

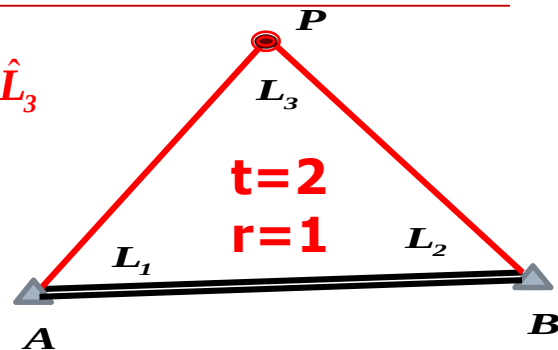
2) 附参条件平差：少量引入 $u=1 < t$ ， $c=r+u=2$

$$\hat{X}_1 = \hat{L}_1 \quad \begin{cases} \hat{L}_1 + \hat{L}_2 + \hat{L}_3 - 180^\circ = 0 \\ \hat{L}_1 - \hat{X}_1 = 0 \end{cases} \quad F_{c1}(\hat{L}_{n1}, \hat{X}_{u1}) = 0$$

3) 间接平差：适量引入 $u=t=2$ ， $c=r+u=3$

$$\begin{aligned} \hat{X}_1 &= \hat{L}_1 \\ \hat{X}_2 &= \hat{L}_2 \end{aligned} \quad \begin{cases} \hat{L}_1 + \hat{L}_2 + \hat{L}_3 - 180^\circ = 0 \\ \hat{L}_1 - \hat{X}_1 = 0 \\ \hat{L}_2 - \hat{X}_2 = 0 \end{cases} \Rightarrow \begin{cases} \hat{L}_1 = \hat{X}_1 \\ \hat{L}_2 = \hat{X}_2 \\ \hat{L}_3 = -\hat{X}_1 - \hat{X}_2 + 180^\circ \end{cases}$$
$$F_{n1}(\hat{L}_{n1}, \hat{X}_{t1}) = 0 \Rightarrow \hat{L}_{n1} = F_{n1}(\hat{X}_{t1})$$

4) 附限间接平差：大量引入 $u > t$ $\hat{L}$ 可用 t 个 $\hat{X}$ 来表示



必要观测的性质：②
必要元素之间函数独立，模型中量都可由必要元素来表示。

§6-1 间接平差原理

1、间接平差原理(先求 $\hat{X}$ ，再求 $\hat{L}$)

(1) 间接平差的函数模型—误差方程

如引入（适量） $u=t$ 个独立参数 $\hat{X}$ ，则函数模型个数 $c=r+t=n$

$$F(\hat{L}_{n1}, \hat{X}_{t1}) = 0 \Rightarrow \hat{L}_{n1} = F(\hat{X}_{t1})$$

$\hat{L}$ 可用 t 个 $\hat{X}$ 来表示

误差方程

$$\begin{cases} \text{非线性: } \hat{L}_{n1} = F(\hat{X}_{t1}) \\ \text{线性: } \hat{L}_{n1} = B_{nt} \hat{X}_{t1} + d_{n1} \end{cases} \xrightarrow{\hat{L} = L + V} \begin{matrix} V_{n1} = B_{nt} \hat{X}_{t1} - l_{n1} \\ \text{改正数误差方程} \end{matrix}$$

$$F(\hat{X}_{t1}) \approx F(X_{t1}^0) + \left(\frac{\partial F}{\partial \hat{X}}\right)_{nt} (\hat{X}_{t1} - X_{t1}^0) \quad B_{nt} = \left(\frac{\partial F}{\partial \hat{X}}\right)_{nt}$$

$$= B_{nt} \hat{X}_{t1} + F_{n1}(X_{t1}^0) \quad l_{n1} = L_{n1} - F_{n1}(X_{t1}^0) = L_{n1} - (B_{nt} X_{t1}^0 + d_{n1})$$

必要观测的性质：
②必要元素之间函数独立，模型中量都可由必要元素来表示。

(2) 间接平差的法方程-函数模型+最小二乘原则

(几何关系)

$$\left. \begin{array}{l} \text{函数模型: } \mathbf{V}_{n1} = \mathbf{B}_{nt} \hat{\mathbf{x}}_{t1} - \mathbf{l}_{n1} \\ \text{平差原则: } \mathbf{V}_{n1}^T \mathbf{P}_{nn} \mathbf{V}_{n1} = \min \end{array} \right\} \Rightarrow \hat{\mathbf{x}}, \mathbf{V} = ?$$

$$\text{随机模型: } \mathbf{D} = \sigma_0^2 \mathbf{P}^{-1} = \sigma_0^2 \mathbf{Q}$$

$\mathbf{V}$ 是 $\hat{\mathbf{x}}$ 的线性函数, $\mathbf{V}^T \mathbf{P} \mathbf{V}$ 是 $\hat{\mathbf{x}}$ 的函数。
现在求 $\mathbf{V}^T \mathbf{P} \mathbf{V} = \min$ 的极值点 $\hat{\mathbf{x}}$ 。

$$\mathbf{V}^T \mathbf{P} \mathbf{V} = \min \text{的必要条件是: } \frac{\partial(\mathbf{V}^T \mathbf{P} \mathbf{V})}{\partial \hat{\mathbf{x}}} = \frac{\partial(\mathbf{V}^T \mathbf{P} \mathbf{V})}{\partial \mathbf{V}} \frac{\partial \mathbf{V}}{\partial \hat{\mathbf{x}}} = 2 \mathbf{V}^T \mathbf{P} \mathbf{B} = 0$$

$$\begin{array}{c} \text{(极值关系)} \\ \uparrow \\ \mathbf{B}_{tn}^T \mathbf{P}_{n1} \mathbf{V}_{n1} = 0 \end{array} \Rightarrow \mathbf{N}_{BB} \hat{\mathbf{x}}_{t1} - \mathbf{W}_{t1} = 0 \quad \text{法方程}$$

$$\begin{array}{c} \text{(几何关系)} \\ \mathbf{V}_{n1} = \mathbf{B}_{nt} \hat{\mathbf{x}}_{t1} - \mathbf{l}_{n1} \end{array} \quad \begin{array}{l} \mathbf{N}_{BB} = \mathbf{B}_{tn}^T \mathbf{P}_{nn} \mathbf{B}_{nt}, \quad \mathbf{W}_{t1} = \mathbf{B}_{tn}^T \mathbf{P}_{nn} \mathbf{l}_{n1} \\ \mathbf{R}(\mathbf{N}_{BB}) = \mathbf{R}(\mathbf{B}^T \mathbf{P} \mathbf{B}) = \mathbf{R}(\mathbf{B}) = t \end{array}$$

$$\mathbf{N}_{BB} \text{为 } t \text{ 阶满秩方阵, 对称可逆。则} \quad \hat{\mathbf{x}}_{t1} = \mathbf{N}_{BB}^{-1} \mathbf{W}_{t1}$$

(3) 间接平差的结果

参数增量 $\hat{\mathbf{x}}$ $\hat{\mathbf{x}}_{t1} = \mathbf{N}_{BB}^{-1} \mathbf{W}$

改正数 $\mathbf{V}$ $\mathbf{V}_{n1} = \mathbf{B}_{nt} \hat{\mathbf{x}}_{t1} - \mathbf{l}_{n1}$

观测值的平差值 $\hat{\mathbf{L}} = \mathbf{L} + \mathbf{V}$

参数平差值 $\hat{\mathbf{X}} = \mathbf{X}^0 + \hat{\mathbf{x}}$

1、间接平差原理（矩阵式-代数式）

误差方程

$$\hat{\mathbf{L}}_{n1} = \mathbf{F}_{n1}(\hat{\mathbf{X}}_{t1})$$

$$\begin{cases} \hat{L}_1 = F_1(\hat{X}_1, \hat{X}_2 \cdots \hat{X}_t) \\ \hat{L}_2 = F_2(\hat{X}_1, \hat{X}_2 \cdots \hat{X}_t) \\ \cdots \\ \hat{L}_n = F_n(\hat{X}_1, \hat{X}_2 \cdots \hat{X}_t) \end{cases}$$

$$\hat{\mathbf{L}}_{n1} = \mathbf{B}_{nt \ t1} \hat{\mathbf{X}}_{t1} + \mathbf{d}_{n1}$$

$$\begin{cases} \hat{L}_1 = b_{11}\hat{X}_1 + b_{12}\hat{X}_2 \cdots + b_{1t}\hat{X}_t + d_1 \\ \hat{L}_2 = b_{21}\hat{X}_1 + b_{22}\hat{X}_2 \cdots + b_{2t}\hat{X}_t + d_2 \\ \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \\ \hat{L}_n = b_{n1}\hat{X}_1 + b_{n2}\hat{X}_2 \cdots + b_{nt}\hat{X}_t + d_n \end{cases}$$

改正数的误差方程

$$\mathbf{V}_{n1} = \mathbf{B}_{nt \ t1} \hat{\mathbf{X}}_{t1} - \mathbf{l}_{n1}$$

$$\begin{cases} v_1 = b_{11}\hat{x}_1 + b_{12}\hat{x}_2 + \cdots + b_{1t}\hat{x}_t - l_1 \\ v_2 = b_{21}\hat{x}_1 + b_{22}\hat{x}_2 + \cdots + b_{2t}\hat{x}_t - l_2 \\ \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \\ v_n = b_{n1}\hat{x}_1 + b_{n2}\hat{x}_2 + \cdots + b_{nt}\hat{x}_t - l_n \end{cases}$$

$$\mathbf{B}_{nt} = \left(\frac{\partial \mathbf{F}_{n1}}{\partial \hat{\mathbf{X}}_{t1}} \right)_0 = \begin{pmatrix} \frac{\partial F_1}{\partial \hat{X}_1} & \frac{\partial F_1}{\partial \hat{X}_2} & \cdots & \frac{\partial F_1}{\partial \hat{X}_t} \\ \frac{\partial F_2}{\partial \hat{X}_1} & \frac{\partial F_2}{\partial \hat{X}_2} & \cdots & \frac{\partial F_2}{\partial \hat{X}_t} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial \hat{X}_1} & \frac{\partial F_n}{\partial \hat{X}_2} & \cdots & \frac{\partial F_n}{\partial \hat{X}_t} \end{pmatrix}_0 = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1t} \\ b_{21} & b_{22} & \cdots & b_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nt} \end{pmatrix}$$

$$\mathbf{l}_{n1} = \mathbf{L}_{n1} - \mathbf{F}_{n1}(\mathbf{X}^0) = \begin{pmatrix} L_1 - F_1(X^0) \\ L_2 - F_2(X^0) \\ \cdots \quad \cdots \\ L_n - F_n(X^0) \end{pmatrix} = \begin{pmatrix} L_1 - b_{11}X_1^0 - \cdots - b_{1t}X_t^0 - d_1 \\ L_2 - b_{21}X_1^0 - \cdots - b_{2t}X_t^0 - d_2 \\ \cdots \quad \cdots \\ L_n - b_{n1}X_1^0 - \cdots - b_{nt}X_t^0 - d_n \end{pmatrix}$$

1、间接平差原理（矩阵式-代数式）

法方程 $N_{BB} \hat{x}_{t1} - W_{t1} = 0$

不相关
观测

$$\left\{ \begin{array}{l} \sum_{i=1}^n p_i b_{i1} b_{i1} \hat{x}_1 + \sum_{i=1}^n p_i b_{i1} b_{i2} \hat{x}_2 \cdots + \sum_{i=1}^n p_i b_{i1} b_{it} \hat{x}_t - w_1 = 0 \\ \sum_{i=1}^n p_i b_{i2} b_{i1} \hat{x}_1 + \sum_{i=1}^n p_i b_{i2} b_{i2} \hat{x}_2 \cdots + \sum_{i=1}^n p_i b_{i2} b_{it} \hat{x}_t - w_2 = 0 \\ \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ \sum_{i=1}^n p_i b_{it} b_{i1} \hat{x}_1 + \sum_{i=1}^n p_i b_{it} b_{i2} \hat{x}_2 \cdots + \sum_{i=1}^n p_i b_{it} b_{it} \hat{x}_t - w_t = 0 \end{array} \right.$$

改正数

$$\mathbf{v}_i = \mathbf{b}_{i1} \hat{\mathbf{x}}_1 + \mathbf{b}_{i2} \hat{\mathbf{x}}_2 + \dots + \mathbf{b}_{it} \hat{\mathbf{x}}_t - l_i$$

L的平差值

$$\hat{\mathbf{L}}_i = \mathbf{L}_i + \mathbf{v}_i$$

$$N_{tt}^{BB} = \mathbf{B}_{tn}^T \mathbf{P}_{nm} \mathbf{B}_{nt} \quad \underline{\underline{\text{不相关观测}}} \quad \begin{pmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1t} & b_{2t} & \cdots & b_{nt} \end{pmatrix} \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1t} \\ b_{21} & b_{22} & \cdots & b_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nt} \end{pmatrix}$$

2、间接平差计算步骤

① 建立函数模型。

立列 n 个误差方程 $\hat{L}=F(\hat{X}_{t1})$ ，线性化
变换为 $V = B\hat{x} - l$ 。

② 建立随机模型。取 σ_0^2 ， $D = \sigma_0^2 Q = \sigma_0^2 P^{-1}$ ，求 P 。

③ 建立法方程，求 $\hat{x}$ 。

$$N_{BB}\hat{x} - W = 0, \hat{x} = N_{BB}^{-1}W,$$

$$\hat{X} = X^0 + \hat{x}。$$

④ 求改正数 $V=B\hat{x} - l$ ，平差值 $\hat{L}=L+V$

⑤ 检验平差计算的正确性。

注意事项：

- a) 准确引入 t 个独立参数；
- b) 误差方程线性化时，角度改正数需用 ρ 统一单位。

弧度与 $^{\circ}$ 的转换

$$\pi \text{弧度} = 180^{\circ}$$

$$\begin{aligned} 1 \text{弧度} &= \frac{180^{\circ}}{3.14} = 57.3^{\circ} = \rho^{\circ} \\ &= 3437' = \rho' \\ &= 206265'' = \rho'' \end{aligned}$$

例1：水准网间接平差

① 建立函数模型

误差方程 $\hat{L}=F(\hat{X})$

变换为 $V = B\hat{x} - l$ 。

② 建立随机模型。

取 σ_0^2 ，求 P

③ 建立法方程，求 $\hat{x}$

$$N_{BB}\hat{x} - W = 0$$

$$\hat{x}=N_{BB}^{-1}W$$

④ 求改正数，平差值

$$V=B\hat{x} - l, \quad \hat{L}=L+V$$

⑤ 检验平差计算的正确性。

测段	1	2	3	4	5
$h_i(m)$	2.410	-2.131	2.945	0.820	-3.222
$S_i(km)$	1.0	2.0	1.0	2.0	2.0

$$H_A=112.145m$$

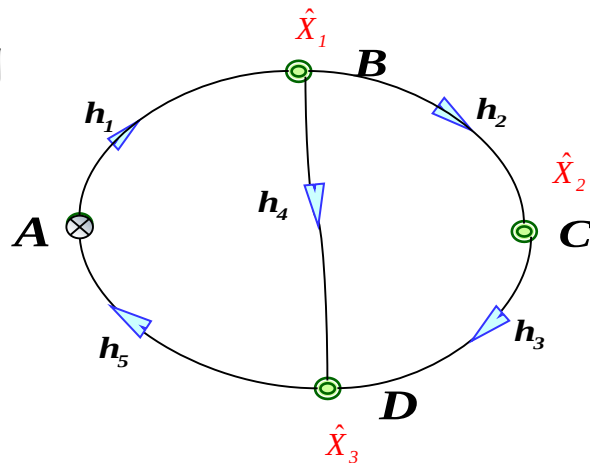
解：（1）观测值个数、必要观测个数
 $n=5, \quad t=3$

引入 $t=3$ 个独立的参数：

$$\hat{X}_1 = \hat{H}_B, \quad X_1^0 = H_A + h_1$$

$$\hat{X}_2 = \hat{H}_C, \quad X_2^0 = H_A + h_1 + h_2$$

$$\hat{X}_3 = \hat{H}_D, \quad X_3^0 = H_A - h_5$$



例1：水准网间接平差

① 建立函数模型

误差方程 $\hat{L}=F(\hat{X})$

变换为 $V = B\hat{x} - l$ 。

② 建立随机模型。

取 σ_0^2 ，求 P

③ 建立法方程, 求 $\hat{x}$

$$N_{BB}\hat{x} - W = 0$$

$$\hat{x} = N_{BB}^{-1}W$$

④ 求改正数, 平差值

$$V = B\hat{x} - l, \quad \hat{L} = L + V$$

⑤ 检验平差计算的准确性。

列误差方程

$$\hat{h}_1 = \hat{X}_1 - H_A$$

$$\hat{h}_2 = -\hat{X}_1 + \hat{X}_3$$

$$\hat{h}_3 = -\hat{X}_2 + \hat{X}_3$$

$$\hat{h}_4 = -\hat{X}_1 + \hat{X}_3$$

$$\hat{h}_5 = -\hat{X}_3 + H_A$$

$$\text{令 } \hat{h}_i = h_i + v_i$$

$$v_1 = \hat{x}_1$$

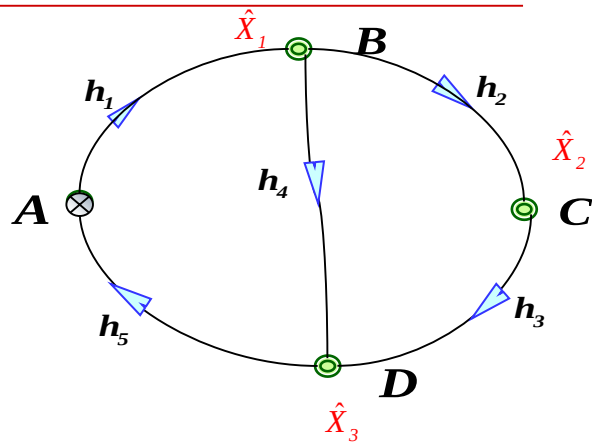
$$v_2 = -\hat{x}_1 + \hat{x}_2$$

$$v_3 = -\hat{x}_2 + \hat{x}_3 - (h_1 + h_2 + h_3 + h_5)$$

$$v_4 = -\hat{x}_1 + \hat{x}_3 - (h_1 + h_4 + h_5)$$

$$v_5 = -\hat{x}_3$$

改正数误差方程



$$B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad l = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 8 \\ 0 \end{pmatrix} \quad (mm)$$

例1：水准网间接平差

① 建立函数模型

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变换为 $V = B\hat{x} - l$ 。

② 建立随机模型。

取 σ_0^2 ，求 P

③ 建立法方程, 求 $\hat{x}$

$$N_{BB}\hat{x} - W = 0$$

$$\hat{x} = N_{BB}^{-1}W$$

④ 求改正数, 平差值

$$V = B\hat{x} - l, \quad \hat{L} = L + V$$

⑤ 检验平差计算的正确性。

(2) 取 $C=1\text{km}$, 1km 水准路线高差观测值为单位权观测值

$$\text{即 } \sigma_0^2 = \sigma_{km}^2$$

高差观测值的权
$$p_i = \frac{C}{S_i} = \frac{1}{S_i}$$

高差观测值的权阵

$$P = \begin{pmatrix} 1 & & & & \\ & 0.5 & & & \\ & & 1 & & \\ & & & 0.5 & \\ & & & & 0.5 \end{pmatrix}$$

高程观测值的权和协因数

测段	1	2	3	4	5
$S_i(\text{km})$	1.0	2.0	1.0	2.0	2.0
$p_i=1/S_i$	1.0	0.5	1.0	0.5	0.5
$Q_{ii}=1/p_i$	1.0	2.0	1.0	2.0	2.0

例1: 水准网间接平差

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② 建立随机模型。

取 σ_0^2 , 求 P

③ 建立法方程, 求 $\hat{x}$

$$N_{BB}\hat{x} - W = 0$$

$$\hat{x} = N_{BB}^{-1}W$$

④ 求改正数, 平差值

$$V = B\hat{x} - l, \quad \hat{L} = L + V$$

⑤ 检验平差计算的准确性。

(3) 建立法方程

$$N_{BB} = B^T P B = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & 0.5 & & & \\ & & 1 & & \\ & & & 0.5 & \\ & & & & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -0.5 & -0.5 \\ -0.5 & 1.5 & -1 \\ -0.5 & -1 & 2 \end{pmatrix}$$

$$W = B^T P l = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & 0.5 & & & \\ & & 1 & & \\ & & & 0.5 & \\ & & & & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix} \text{ (mm)}$$

$$\begin{pmatrix} 2 & -0.5 & -0.5 \\ -0.5 & 1.5 & -1 \\ -0.5 & -1 & 2 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} - \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix} = 0$$

$$\text{解得} \quad \hat{x} = N_{BB}^{-1}W = (-1.3 \ 0.0 \ 2.7)^T \text{ (mm)}$$

例1：水准网间接平差

① 建立函数模型

误差方程 $\hat{L}=F(\hat{X})$
变换为 $V = B\hat{x} - l$ 。

② 建立随机模型。

取 σ_0^2 ，求 P

③ 建立法方程, 求 $\hat{x}$

$$N_{BB}\hat{x} - W = 0$$

$$\hat{x} = N_{BB}^{-1}W$$

④ 求改正数, 平差值

$$V = B\hat{x} - l, \quad \hat{L} = L + V$$

⑤ 检验平差计算的准确性。

(4) 求观测值改正数、平差值

$$V = B\hat{x} - l = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1.3 \\ 0.0 \\ 2.7 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} -1.3 \\ 1.3 \\ 0.7 \\ -4.0 \\ -2.7 \end{pmatrix} (mm)$$

$$\hat{h} = h + V = (2.4087 \quad -2.1297 \quad 2.9457 \quad 0.8160 \quad -3.2247)^T (m)$$

(5) 将平差值代入误差方程，方程左右相等，计算正确。

以上是本节课的内容。

同学们，下节课再见！

第六章 间接平差

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§6-2 间接平差精度评价

§6-3 误差方程类型

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§6-2 间接平差精度评价

测量平差的任务: 1) 求 $\mu=\tilde{L}$ 最优估值 $\hat{L}$, 参数 $\hat{X}$;
2) 求 σ_0^2 的估计 $\hat{\sigma}_0^2$, 评价 L 、 $\hat{L}$ 、 $\hat{X}$ 、 $\varphi(\hat{L}, \hat{X})$ 的精度。

下面从3方面进行

- 1、单位权方差 σ_0^2 的估计
- 2、平差基本变量的协因数阵 Q_{ZZ} , $D_{ZZ} = \hat{\sigma}_0^2 Q_{ZZ}$
- 3、平差值函数 $\varphi(\hat{L}, \hat{X})$ 的方差估值

§6-2 间接平差精度评价

1、单位权方差 σ_0^2 的估计

单位权方差 σ_0^2 的无偏估计量为

$$\hat{\sigma}_0^2 = \frac{V^T P V}{r} \underline{\text{不相关观测}} \frac{1}{r} [P V V] = \frac{1}{r} \sum_{i=1}^n p_i v_i^2$$

$$\begin{aligned} V^T P V &= (B\hat{x} - l)^T P V = \hat{x}^T \underline{B^T P V} - l^T P V \\ &= -l^T P (B\hat{x} - l) \\ &= l^T P l - (B^T P l)^T \hat{x} \end{aligned}$$

$$P_{\text{不相关}} = \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & \ddots \\ & & & 0 \\ & & & & p_n \end{pmatrix}$$

r —多余观测数； P —观测值权阵， V —改正数
观测值的方差估值为

$$\hat{\sigma}_i^2 = \frac{\hat{\sigma}_0^2}{p_i}$$

§6-2 间接平差精度评价

2、间接平差基本变量的协因数阵 Q_{ZZ}

$$Z = \begin{pmatrix} L \\ n1 \\ \hat{X} \\ t1 \\ V \\ n1 \\ \hat{L} \\ n1 \end{pmatrix} \quad \begin{aligned} L &= I \ L \\ \hat{X} &= X^0 + \hat{x} = X^0 + N_{BB}^{-1} B^T P l \\ V &= B\hat{x} - l = (BN_{BB}^{-1} B^T P - I) l \\ \hat{L} &= L + V \end{aligned}$$

$$Z = \begin{pmatrix} L \\ \hat{X} \\ V \\ \hat{L} \end{pmatrix} \quad \begin{pmatrix} L \\ \hat{X} \\ V \\ \hat{L} \end{pmatrix} = Z \quad Q_{ZZ} = \begin{pmatrix} Q_{LL} & Q_{L\hat{X}} & Q_{LV} & Q_{L\hat{L}} \\ Q_{\hat{X}L} & Q_{\hat{X}\hat{X}} & Q_{\hat{X}V} & Q_{\hat{X}\hat{L}} \\ Q_{VL} & Q_{V\hat{X}} & Q_{VV} & Q_{V\hat{L}} \\ Q_{\hat{L}L} & Q_{\hat{L}\hat{X}} & Q_{\hat{L}V} & Q_{\hat{L}\hat{L}} \end{pmatrix}$$

$$Q_{LL} = Q_{ll} = Q \quad l = L - F(X^0) = L - (BX^0 + d)$$

$$Q_{L\hat{X}} = I Q (N_{BB}^{-1} B^T P)^T = B N_{BB}^{-1}$$

$$Q_{LV} = I Q (B N_{BB}^{-1} B^T P - I)^T = B N_{BB}^{-1} B^T - Q$$

$$Q_{L\hat{L}} = Q_{LL} + Q_{LV} = Q + B N_{BB}^{-1} B^T - Q = B N_{BB}^{-1} B^T$$

$$Q_{\hat{X}\hat{X}} = Q_{\hat{x}\hat{x}} = N_{BB}^{-1} B^T P Q (P B N_{BB}^{-1}) = N_{BB}^{-1}$$

$$\begin{aligned} Q_{\hat{X}V} &= N_{BB}^{-1} B^T P Q (B N_{BB}^{-1} B^T P - I)^T \\ &= N_{BB}^{-1} \underline{B^T P B} N_{BB}^{-1} B^T - N_{BB}^{-1} B^T = 0 \end{aligned}$$

$$Q_{\hat{X}\hat{L}} = Q_{\hat{X}L} + Q_{\hat{X}V} = Q_{\hat{X}L} = N_{BB}^{-1} B^T$$

$$N_{BB} = B^T P B$$

$$\begin{aligned} Q_{\hat{L}Y} &= Q_{LY} + Q_{VY} \\ Q_{Y\hat{L}} &= Q_{YL} + Q_{YV} \end{aligned}$$

§6-2 间接平差精度评价

2、间接平差基本变量的协因数阵 Q_{ZZ}

$$Z = \begin{pmatrix} L \\ n1 \\ \hat{X} \\ t1 \\ V \\ n1 \\ \hat{L} \\ n1 \end{pmatrix}$$

$$L = I \quad L$$

$$\hat{X} = X^0 + \hat{x} = X^0 + N_{BB}^{-1} B^T P l$$

$$V = B\hat{x} - l = (BN_{BB}^{-1} B^T P - I) l$$

$$\hat{L} = L + V$$

$$Z = \begin{pmatrix} L \\ \hat{X} \\ V \\ \hat{L} \end{pmatrix} \quad \begin{pmatrix} L \\ \hat{X} \\ V \\ \hat{L} \end{pmatrix} = Z \quad Q_{ZZ} = \begin{pmatrix} Q_{LL} & Q_{L\hat{X}} & Q_{LV} & Q_{L\hat{L}} \\ Q_{\hat{X}L} & Q_{\hat{X}\hat{X}} & Q_{\hat{X}V} & Q_{\hat{X}\hat{L}} \\ Q_{VL} & Q_{V\hat{X}} & Q_{VV} & Q_{V\hat{L}} \\ Q_{\hat{L}L} & Q_{\hat{L}\hat{X}} & Q_{\hat{L}V} & Q_{\hat{L}\hat{L}} \end{pmatrix}$$

$$Q_{LL} = Q_{ll} = Q \quad Q_{\hat{X}\hat{X}} = Q_{\hat{x}\hat{x}}$$

$$Q_{VV} = (B \quad -I) \begin{pmatrix} Q_{\hat{x}\hat{x}} & Q_{\hat{x}L} \\ Q_{L\hat{x}} & Q_{LL} \end{pmatrix} \begin{pmatrix} B^T \\ -I \end{pmatrix} = (BQ_{\hat{x}\hat{x}} - Q_{L\hat{x}} \quad BQ_{\hat{x}L} - Q) \begin{pmatrix} B^T \\ -I \end{pmatrix}$$

$$= BQ_{\hat{x}\hat{x}} B^T - Q_{L\hat{x}} B^T - BQ_{\hat{x}L} + Q$$

$$= BN_{BB}^{-1} B^T - BN_{BB}^{-1} B^T - BN_{BB}^{-1} B^T + Q$$

$$= Q - BN_{BB}^{-1} B^T$$

$$Q_{VV} = (BN_{BB}^{-1} B^T P - I) Q (PBN_{BB}^{-1} B^T - I)$$

$$= (BN_{BB}^{-1} B^T P - I) (BN_{BB}^{-1} B^T - Q)$$

$$= BN_{BB}^{-1} B^T - BN_{BB}^{-1} B^T - BN_{BB}^{-1} B^T + Q$$

$$= Q - BN_{BB}^{-1} B^T$$

$$Q_{V\hat{L}} = Q_{VL} + Q_{VV}$$

$$= BN_{BB}^{-1} B^T - Q + Q - BN_{BB}^{-1} B^T$$

$$= 0$$

$$Q_{\hat{L}\hat{L}} = Q_{LL} + Q_{LV} + Q_{VL} + Q_{VV} = Q - Q_{N_{BB}^{VV}} = B^T P B$$

$$= BN_{BB}^{-1} B^T$$

间接平差基本变量的协因数表

	L	$\hat{X}$	V	$\hat{L}$
L	Q	BN_{BB}^{-1}	$BN_{BB}^{-1}B^T-Q$	$BN_{BB}^{-1}B^T$
$\hat{X}$	$N_{BB}^{-1}B^T$	N_{BB}^{-1}	O	$N_{BB}^{-1}B^T$
V	$BN_{BB}^{-1}B^T-Q$	O	$Q-BN_{BB}^{-1}B^T$	O
$\hat{L}$	$BN_{BB}^{-1}B^T$	BN_{BB}^{-1}	O	$BN_{BB}^{-1}B^T$

$\hat{L}$ 与 V 不相关,
 $\hat{X}$ 与 V 不相关

Z的方差阵 $D_{ZZ} = \hat{\sigma}_0^2 Q_{ZZ}$

§6-2 间接平差精度评价

3、平差值函数的方差估计

参数 $\hat{X}$ 的 m 个函数 $\varphi(\hat{X})$

$$\varphi = \varphi(\hat{X}) = \begin{pmatrix} \varphi_1(\hat{X}) \\ \varphi_2(\hat{X}) \\ \dots \\ \varphi_m(\hat{X}) \end{pmatrix}$$

对 $\varphi(\hat{X})$ 求全微分，得系数矩阵

$$d\varphi = \left(\frac{\partial \varphi}{\partial \hat{X}} \right)_0 d\hat{X} = f_{x_{mt}}^T d\hat{X}_{t1}$$

由协因数传播定理，得

$$Q_{\varphi\varphi} = f_x^T Q_{\hat{X}\hat{X}} f_x = f_x^T N_{BB}^{-1} f_x$$

求各函数的方差估值

$$\hat{\sigma}_{\varphi_i}^2 = \hat{\sigma}_0^2 Q_{\varphi_i \varphi_i}$$

$$i = 1, 2, \dots, m$$

$$f_{x_{mt}}^T = \left(\frac{\partial \varphi}{\partial \hat{X}} \right)_0 = \begin{pmatrix} \frac{\partial \varphi_1}{\partial \hat{X}_1} & \frac{\partial \varphi_1}{\partial \hat{X}_2} \dots & \frac{\partial \varphi_1}{\partial \hat{X}_t} \\ \frac{\partial \varphi_2}{\partial \hat{X}_1} & \frac{\partial \varphi_2}{\partial \hat{X}_2} \dots & \frac{\partial \varphi_2}{\partial \hat{X}_t} \\ \vdots & \vdots & \vdots \\ \frac{\partial \varphi_m}{\partial \hat{X}_1} & \frac{\partial \varphi_m}{\partial \hat{X}_2} \dots & \frac{\partial \varphi_m}{\partial \hat{X}_t} \end{pmatrix}_0$$

以上是本节课的内容。

同学们，下节课再见！

第六章 间接平差

§6-1 间接平差原理

§6-2 间接平差精度评价

§6-3 误差方程类型

§6-4 间接平差实例

§6-3 误差方程类型

间接平差的函数模型称为**误差方程**，观测类型决定误差方程类型。

- 1、水准网间接平差的误差方程
 - 2、测方向间接平差的误差方程
 - 3、测角网坐标平差的误差方程
 - 4、测边网坐标平差的误差方程
 - 5、边角网（导线网）坐标平差的误差方程
 - 6、GPS网控制网平差的误差方程
-

1、水准网间接平差的误差方程

观测值：高差 $h_i, i=1,2\dots n$

平差值：高差 $\hat{h}_i, i=1,2\dots n$

引入独立参数：高程平差值 $\hat{X}_i, i=1,2\dots t$

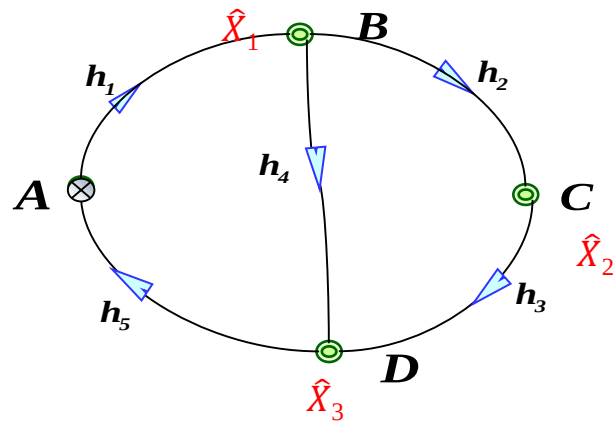
示例中， $n=5, t=3$

引入独立参数，取近似值：

$$\hat{X}_1 = \hat{H}_B, \quad X_1^0 = H_A + h_1$$

$$\hat{X}_2 = \hat{H}_C, \quad X_2^0 = H_A + h_1 + h_2$$

$$\hat{X}_3 = \hat{H}_D, \quad X_3^0 = H_A - h_5$$



$$\begin{cases} \hat{h}_1 = \hat{X}_1 - H_A \\ \hat{h}_2 = -\hat{X}_1 + \hat{X}_2 \\ \hat{h}_3 = -\hat{X}_2 + \hat{X}_3 \\ \hat{h}_4 = -\hat{X}_1 + \hat{X}_3 \\ \hat{h}_5 = -\hat{X}_3 + H_A \end{cases} \quad \hat{X}_i = X_i^0 + \hat{x}_i \quad \hat{h}_i = h_i + v_i$$

$$\hat{\mathbf{L}} = \mathbf{F}(\hat{\mathbf{X}})$$

$$\begin{cases} v_1 = \hat{x}_1 \\ v_2 = -\hat{x}_1 + \hat{x}_2 \\ v_3 = -\hat{x}_2 + \hat{x}_3 - (h_1 + h_2 + h_3 + h_5) \\ v_4 = -\hat{x}_1 + \hat{x}_3 - (h_1 + h_4 + h_5) \\ v_5 = -\hat{x}_3 \end{cases}$$

$$\mathbf{V} = \mathbf{B} \hat{\mathbf{x}} - \mathbf{l}$$

$$\mathbf{B}_{53} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathbf{l}_{51} = \begin{pmatrix} 0 \\ 0 \\ h_1 + h_2 + h_3 + h_5 \\ h_1 + h_4 + h_5 \\ 0 \end{pmatrix}$$

2、测方向间接平差的误差方程

方向观测网，方向观测值 L_{ij} ，方向平差值 $\hat{L}_{ij}$ 。引入独立参数：坐标平差值 $\hat{x}$ 和 i 测站的定向角（零方向方位角） $\hat{z}_i$

(1) 建立方向平差值关于方位角的误差方程

$$\hat{L}_{ij} = \hat{\alpha}_{ij} - \hat{z}_i$$

$$\hat{L}_{ij} = L_{ij} + v_{ij}$$

$$\hat{z} = z_i^0 + \hat{z}_i$$

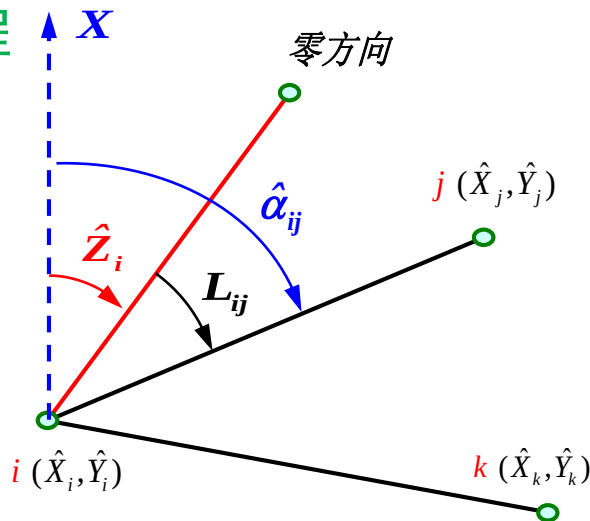
$$\hat{\alpha}_{ij} = \alpha_{ij}^0 + \delta\hat{\alpha}_{ij}$$

$$v_{ij} = \hat{\alpha}_{ij} - \hat{z}_i - L_{ij}$$

$$= \delta\hat{\alpha}_{ij} - \hat{z}_i - (L_{ij} + z_i^0 - \alpha_{ij}^0)$$

$$= \delta\hat{\alpha}_{ij} - \hat{z}_i - l_{ij}$$

$$l_{ij} = L_{ij} - (\alpha_{ij}^0 - z_i^0)$$



(2) 建立方位角与坐标的关系式

$$(\arctan \mathbf{x})' = \frac{1}{1 + \mathbf{x}^2}$$

$$\begin{aligned}\hat{\alpha}_{ij} &= \arctan \frac{\hat{\mathbf{Y}}_j - \hat{\mathbf{Y}}_i}{\hat{\mathbf{X}}_j - \hat{\mathbf{X}}_i} \\ &= \arctan \frac{\mathbf{Y}_j^0 - \mathbf{Y}_i^0}{\mathbf{X}_j^0 - \mathbf{X}_i^0} + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{X}}} \right)_o (\hat{\mathbf{X}} - \mathbf{X}^0) \\ &= \alpha_{ij}^0 + \delta \hat{\alpha}_{ij}\end{aligned}$$

$$\left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{X}}_i} \right)_o = \frac{1}{1 + \left(\frac{\hat{\mathbf{Y}}_j - \hat{\mathbf{Y}}_i}{\hat{\mathbf{X}}_j - \hat{\mathbf{X}}_i} \right)^2} \cdot \left[- \frac{\hat{\mathbf{Y}}_j - \hat{\mathbf{Y}}_i}{(\hat{\mathbf{X}}_j - \hat{\mathbf{X}}_i)^2} \right]_o \cdot (-1)$$

$$= \frac{\hat{\mathbf{Y}}_j - \hat{\mathbf{Y}}_i}{(\hat{\mathbf{X}}_j - \hat{\mathbf{X}}_i)^2 + (\hat{\mathbf{Y}}_j - \hat{\mathbf{Y}}_i)^2} \bigg|_o = \frac{\Delta \mathbf{Y}_{ij}^0}{(\mathbf{S}_{ij}^0)^2}$$

$$\left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{Y}}_i} \right)_o = - \frac{\Delta \mathbf{X}_{ij}^0}{(\mathbf{S}_{ij}^0)^2} \quad \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{X}}_j} \right)_o = - \frac{\Delta \mathbf{Y}_{ij}^0}{(\mathbf{S}_{ij}^0)^2} \quad \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{Y}}_j} \right)_o = \frac{\Delta \mathbf{X}_{ij}^0}{(\mathbf{S}_{ij}^0)^2}$$

$$\begin{aligned}\delta \hat{\alpha}_{ij} &= \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{X}}_i} \right)_o (\hat{\mathbf{X}}_i - \mathbf{X}_i^0) + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{Y}}_i} \right)_o (\hat{\mathbf{Y}}_i - \mathbf{Y}_i^0) + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{X}}_j} \right)_o (\hat{\mathbf{X}}_j - \mathbf{X}_j^0) + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{Y}}_j} \right)_o (\hat{\mathbf{Y}}_j - \mathbf{Y}_j^0) \\ &= \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{X}}_i} \right)_o \hat{\mathbf{x}}_i + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{Y}}_i} \right)_o \hat{\mathbf{y}}_i + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{X}}_j} \right)_o \hat{\mathbf{x}}_j + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{\mathbf{Y}}_j} \right)_o \hat{\mathbf{y}}_j \\ &= \frac{\rho'' \Delta \mathbf{Y}_{ij}^0}{(\mathbf{S}_{ij}^0)^2} \hat{\mathbf{x}}_i - \frac{\rho'' \Delta \mathbf{X}_{ij}^0}{(\mathbf{S}_{ij}^0)^2} \hat{\mathbf{y}}_i - \frac{\rho'' \Delta \mathbf{Y}_{ij}^0}{(\mathbf{S}_{ij}^0)^2} \hat{\mathbf{x}}_j + \frac{\rho'' \Delta \mathbf{X}_{ij}^0}{(\mathbf{S}_{ij}^0)^2} \hat{\mathbf{y}}_j\end{aligned}$$

(3) 建立方向改正数关于坐标的误差方程

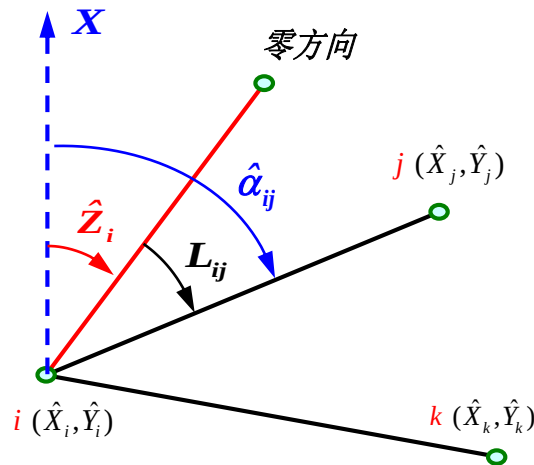
$$\begin{aligned}
 v_{ij} &= \hat{\alpha}_{ij} - \hat{Z}_i - L_{ij} \\
 &= \delta\hat{\alpha}_{ij} - \hat{Z}_i - l_{ij} \\
 &= -\hat{Z}_i + \frac{\rho''\Delta Y_{ij}^0}{(S_{ij}^0)^2} \hat{x}_i - \frac{\rho''\Delta X_{ij}^0}{(S_{ij}^0)^2} \hat{y}_i - \frac{\rho''\Delta Y_{ij}^0}{(S_{ij}^0)^2} \hat{x}_j + \frac{\rho''\Delta X_{ij}^0}{(S_{ij}^0)^2} \hat{y}_j - l_{ij} \\
 &= -\hat{Z}_i + a_{ij} \hat{x}_i - b_{ij} \hat{y}_i - a_{ij} \hat{x}_j + b_{ij} \hat{y}_j - l_{ij} \\
 a_{ij} &= \frac{\rho''\Delta Y_{ij}^0}{(S_{ij}^0)^2}, \quad b_{ij} = \frac{\rho''\Delta X_{ij}^0}{(S_{ij}^0)^2} \quad l_{ij} = L_{ij} - (\alpha_{ij}^0 - Z_i^0)
 \end{aligned}$$

特别地

i点已知时, $\hat{x}_i=0, \hat{y}_i=0$ $v_{ij} = -\hat{Z}_i - a_{ij} \hat{x}_j + b_{ij} \hat{y}_j - l_{ij}$

j点已知时, $\hat{x}_j=0, \hat{y}_j=0$ $v_{ij} = -\hat{Z}_i + a_{ij} \hat{x}_i - b_{ij} \hat{y}_i - l_{ij}$

i、j点已知时, $v_{ij} = -\hat{Z}_i - l_{ij}$



方向误差方程的规律性

$$v_{ij} = -\hat{z}_i + a_{ij}\hat{x}_i - b_{ij}\hat{y}_i - a_{ij}\hat{x}_j + b_{ij}\hat{y}_j - l_{ij}$$

1) 若 i 、 j 均为待定点，坐标改正数系数符号相反，数值相等。

$$\delta\hat{\alpha}_{ji} = \delta\hat{\alpha}_{ij}$$

$$a_{ji} = -a_{ij}, \quad b_{ji} = -b_{ij}$$

2) 坐标改正数单位的选取

取 $dm: /10$

$$v_{ij} = -\hat{z}_i + \frac{\rho'' \Delta Y_{ij}^0}{10 \cdot (S_{ij}^0)^2} \hat{x}_i - \frac{\rho'' \Delta X_{ij}^0}{10 \cdot (S_{ij}^0)^2} \hat{y}_i - \frac{\rho'' \Delta Y_{ij}^0}{10 \cdot (S_{ij}^0)^2} \hat{x}_j + \frac{\rho'' \Delta X_{ij}^0}{10 \cdot (S_{ij}^0)^2} \hat{y}_j - l_{ij}$$

取 $cm: /100$

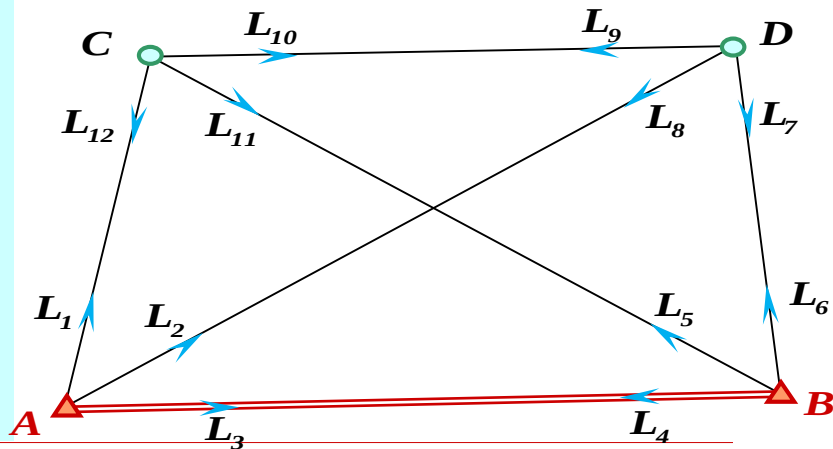
$$v_{ij} = -\hat{z}_i + \frac{\rho'' \Delta Y_{ij}^0}{100 \cdot (S_{ij}^0)^2} \hat{x}_i - \frac{\rho'' \Delta X_{ij}^0}{100 \cdot (S_{ij}^0)^2} \hat{y}_i - \frac{\rho'' \Delta Y_{ij}^0}{100 \cdot (S_{ij}^0)^2} \hat{x}_j + \frac{\rho'' \Delta X_{ij}^0}{100 \cdot (S_{ij}^0)^2} \hat{y}_j - l_{ij}$$

例:试列出图示网间接平差

1) 如图方向观测网参数选择

$$n=12, t=4+4=8$$

$$\hat{\underset{8,1}{\mathbf{X}}} = \begin{pmatrix} \hat{\mathbf{Z}}_A \\ \hat{\mathbf{Z}}_B \\ \hat{\mathbf{Z}}_C \\ \hat{\mathbf{Z}}_D \\ \hat{\mathbf{X}}_C \\ \hat{\mathbf{Y}}_C \\ \hat{\mathbf{X}}_D \\ \hat{\mathbf{Y}}_D \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_A^0 \\ \mathbf{Z}_B^0 \\ \mathbf{Z}_C^0 \\ \mathbf{Z}_D^0 \\ \mathbf{X}_C^0 \\ \mathbf{Y}_C^0 \\ \mathbf{X}_D^0 \\ \mathbf{Y}_D^0 \end{pmatrix} + \begin{pmatrix} \hat{\mathbf{z}}_A \\ \hat{\mathbf{z}}_B \\ \hat{\mathbf{z}}_C \\ \hat{\mathbf{z}}_D \\ \hat{\mathbf{x}}_C \\ \hat{\mathbf{y}}_C \\ \hat{\mathbf{x}}_D \\ \hat{\mathbf{y}}_D \end{pmatrix} = \underset{8,1}{\mathbf{X}}^0 + \underset{8,1}{\hat{\mathbf{x}}}$$



3、测角网坐标平差的误差方程

$$\hat{L}_{n1} = F(\hat{X}_{n1, t1})$$

$$V = B \hat{x} - l$$

角度观测值: $L_i, i=1, 2 \dots n$

角度平差值: $\hat{L}_i, i=1, 2 \dots n$

引入独立参数: 坐标平差值 $\hat{X}$

$i(\hat{X}_i, \hat{Y}_i) \quad j(\hat{X}_j, \hat{Y}_j) \quad k(\hat{X}_k, \hat{Y}_k)$

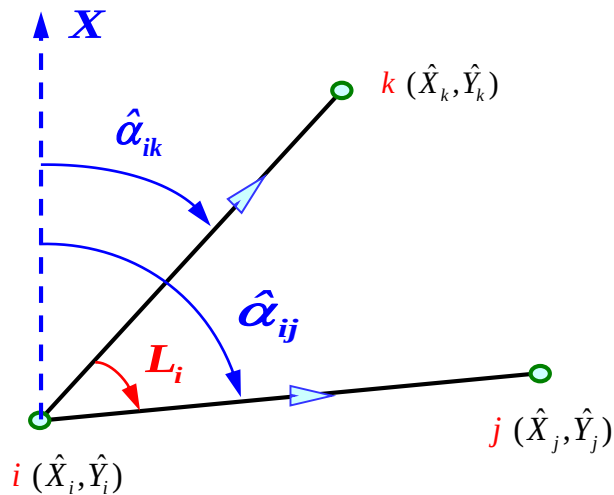
误差方程的建立步骤思路:

(1) 建立角度平差值关于方位角的误差方程

(2) 建立方位角平差值关于坐标的关系式, 线性化

(3) 建立角度改正数关于坐标的误差方程

$$\begin{aligned}\hat{L}_i &= \hat{\alpha}_{ij} - \hat{\alpha}_{ik} \\ \hat{\alpha}_{ij} &= \arctan \frac{\hat{Y}_j - \hat{Y}_i}{\hat{X}_j - \hat{X}_i} \\ v_i &= B_i \hat{x} - l_i\end{aligned}$$



(1) 建立角度平差值关于方位角的误差方程

$$\hat{L}_i = \hat{\alpha}_{ij} - \hat{\alpha}_{ik}$$

$$\hat{L}_i = L_i + v_i$$

$$\hat{\alpha}_{ij} = \alpha_{ij}^0 + \delta\hat{\alpha}_{ij}$$

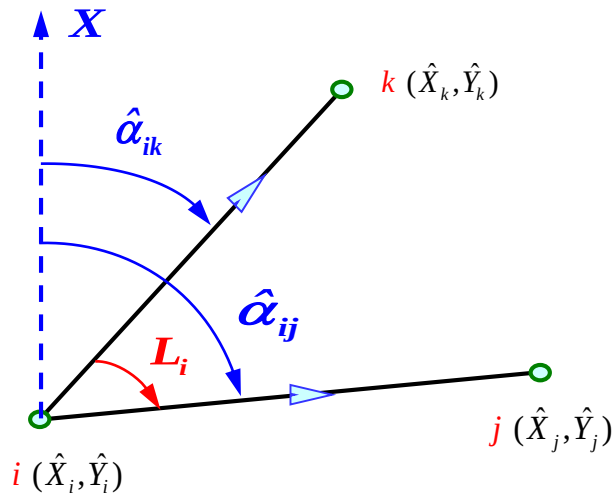
$$\hat{\alpha}_{ik} = \alpha_{ik}^0 + \delta\hat{\alpha}_{ik}$$

$$v_i = \hat{\alpha}_{ij} - \hat{\alpha}_{ik} - L_i$$

$$= \delta\hat{\alpha}_{ij} - \delta\hat{\alpha}_{ik} - (L_i - (\alpha_{ij}^0 - \alpha_{ik}^0))$$

$$= \delta\hat{\alpha}_{ij} - \delta\hat{\alpha}_{ik} - l_i$$

$$l_i = L_i - (\alpha_{ik}^0 - \alpha_{ij}^0)$$



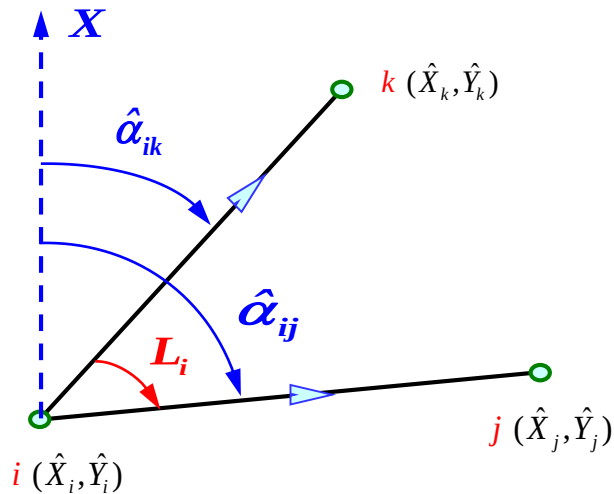
(2) 建立方位角平差值与坐标的关系式

$$\begin{aligned}
 \hat{\alpha}_{ij} &= \arctan \frac{\hat{Y}_j - \hat{Y}_i}{\hat{X}_j - \hat{X}_i} \\
 &= \arctan \frac{Y_j^0 - Y_i^0}{X_j^0 - X_i^0} + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{X}} \right)_0 (\hat{X} - X^0) \\
 &= \alpha_{ij}^0 + \delta \hat{\alpha}_{ij} \\
 \delta \hat{\alpha}_{ij} &= \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{X}_i} \right)_0 \hat{x}_i + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{Y}_i} \right)_0 \hat{y}_i + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{X}_j} \right)_0 \hat{x}_j + \left(\frac{\partial \hat{\alpha}_{ij}}{\partial \hat{Y}_j} \right)_0 \hat{y}_j \\
 &= \frac{\rho'' \Delta Y_{ij}^0}{(S_{ij}^0)^2} \hat{x}_i - \frac{\rho'' \Delta X_{ij}^0}{(S_{ij}^0)^2} \hat{y}_i - \frac{\rho'' \Delta Y_{ij}^0}{(S_{ij}^0)^2} \hat{x}_j + \frac{\rho'' \Delta X_{ij}^0}{(S_{ij}^0)^2} \hat{y}_j
 \end{aligned}$$

$$\delta \hat{\alpha}_{ij} = a_{ij} \hat{x}_i - b_{ij} \hat{y}_i - a_{ij} \hat{x}_j + b_{ij} \hat{y}_j$$

$$\delta \hat{\alpha}_{ik} = a_{ik} \hat{x}_i - b_{ik} \hat{y}_i - a_{ik} \hat{x}_k + b_{ik} \hat{y}_k$$

$$a_{ij} = \frac{\rho'' \Delta Y_{ij}^0}{(S_{ij}^0)^2}, \quad b_{ij} = \frac{\rho'' \Delta X_{ij}^0}{(S_{ij}^0)^2}, \quad a_{ik} = \frac{\rho'' \Delta Y_{ik}^0}{(S_{ik}^0)^2}, \quad b_{ik} = \frac{\rho'' \Delta X_{ik}^0}{(S_{ik}^0)^2}$$



(3) 建立角度改正数关于坐标的误差方程

$$\delta\hat{\alpha}_{ij} = a_{ij}\hat{x}_i - b_{ij}\hat{y}_i - a_{ij}\hat{x}_j + b_{ij}\hat{y}_j$$

$$\delta\hat{\alpha}_{ik} = a_{ik}\hat{x}_i - b_{ik}\hat{y}_i - a_{ik}\hat{x}_k + b_{ik}\hat{y}_k$$

$$\hat{L}_i = \hat{\alpha}_{ij} - \hat{\alpha}_{ik}$$

$$v_i = \delta\hat{\alpha}_{ij} - \delta\hat{\alpha}_{ik} - l_i$$

$$= (a_{ij} - a_{ik})\hat{x}_i + (-b_{ij} + b_{ik})\hat{y}_i - a_{ij}\hat{x}_j + b_{ij}\hat{y}_j + a_{ik}\hat{x}_k - b_{ik}\hat{y}_k - l_i$$

$$l_i = L_i - (\alpha_{ik}^0 - \alpha_{ij}^0)$$

特别地

i点已知时, $\hat{x}_i=0$, $\hat{y}_i=0$

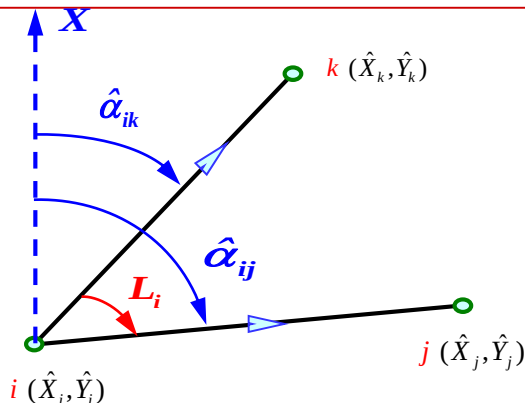
$$v_i = -a_{ij}\hat{x}_j + b_{ij}\hat{y}_j + a_{ik}\hat{x}_k - b_{ik}\hat{y}_k - l_i$$

j点已知时, $\hat{x}_j=0$, $\hat{y}_j=0$

$$v_i = (a_{ij} - a_{ik})\hat{x}_i + (-b_{ij} + b_{ik})\hat{y}_i + a_{ik}\hat{x}_k - b_{ik}\hat{y}_k - l_i$$

i、j点已知时,

$$v_i = a_{ik}\hat{x}_k - b_{ik}\hat{y}_k - l_i$$



列立角度观测值误差方程的步骤

- (1) 选取待定点的坐标作为未知参数。
- (2) 计算参数的近似值，边长、方位角的近似值。
- (3) 计算误差方程各系数项和常数项。
- (4) 列出线性化误差方程。

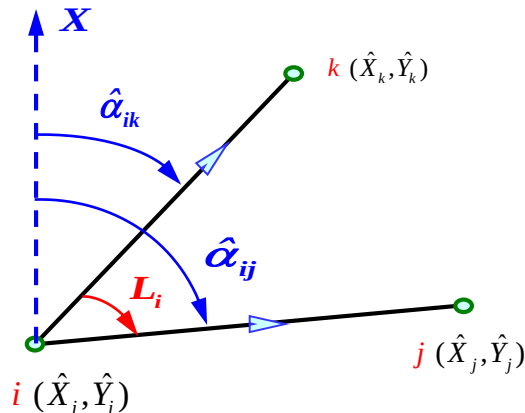
系数 a, b $a_{ij} = \frac{\rho'' \Delta Y_{ij}^0}{(S_{ij}^0)^2}, \quad b_{ij} = \frac{\rho'' \Delta X_{ij}^0}{(S_{ij}^0)^2} \quad a_{ik} = \frac{\rho'' \Delta Y_{ik}^0}{(S_{ik}^0)^2}, \quad b_{ik} = \frac{\rho'' \Delta X_{ik}^0}{(S_{ik}^0)^2}$

常数 l $l_i = L_i - (\alpha_{ik}^0 - \alpha_{ij}^0) \quad a_{ji} = -a_{ij}, \quad b_{ji} = -b_{ij}$

误差方程 $v_i = (a_{ij} - a_{ik}) \hat{x}_i + (-b_{ij} + b_{ik}) \hat{y}_i - a_{ij} \hat{x}_j + b_{ij} \hat{y}_j + a_{ik} \hat{x}_k - b_{ik} \hat{y}_k - l_i$

$$\alpha_{ij}^0 = \arctan \frac{Y_j^0 - Y_i^0}{X_j^0 - X_i^0} \quad \alpha_{ik}^0 = \arctan \frac{Y_k^0 - Y_i^0}{X_k^0 - X_i^0}$$

若坐标改正数取 **cm**，则系数 **a、b** 计算式需除以 **100**



以上是本节课的内容。

同学们，下节课再见！

§6-3 误差方程类型

间接平差的函数模型称为**误差方程**，观测类型决定误差方程类型。

- 1、水准网间接平差的误差方程
 - 2、测方向间接平差的误差方程
 - 3、测角网坐标平差的误差方程
 - 4、测边网坐标平差的误差方程
 - 5、边角网（导线网）坐标平差的误差方程
 - 6、GPS网控制网平差的误差方程
-

4、测边网坐标平差的误差方程

$$\hat{\mathbf{L}}_{n1} = F(\hat{\mathbf{X}}_{n1, t1})$$

$$\mathbf{V}_{n1} = \mathbf{B}_{nt, t1} \hat{\mathbf{x}}_{n1} - \mathbf{l}_{n1}$$

边长观测值: L_{jk} , 角度平差值: $\hat{L}_{jk}$

引入独立参数: 坐标平差值 $j(\hat{X}_j, \hat{Y}_j)$ $k(\hat{X}_k, \hat{Y}_k)$

(1) 边长平差值误差方程

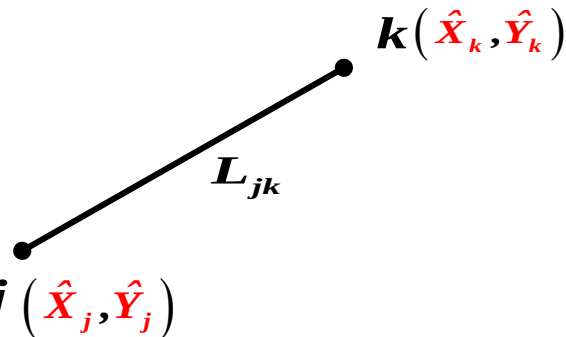
设边长函数

$$\hat{L}_{jk} = \sqrt{(\hat{\mathbf{X}}_k - \hat{\mathbf{X}}_j)^2 + (\hat{\mathbf{Y}}_k - \hat{\mathbf{Y}}_j)^2} = \hat{\mathbf{S}}_{jk} \quad j(\hat{\mathbf{X}}_j, \hat{\mathbf{Y}}_j) \quad k(\hat{\mathbf{X}}_k, \hat{\mathbf{Y}}_k)$$

$$\text{令} \quad \hat{L}_{jk} = L_{jk} + v_{jk}, \quad \hat{\mathbf{S}}_{jk} = \mathbf{S}_{jk}^0 + \hat{\mathbf{s}}_{jk}$$

$$\text{故} \quad v_{jk} = \hat{\mathbf{s}}_{jk} - (L_{jk} - \mathbf{S}_{jk}^0) = \hat{\mathbf{s}}_{jk} - l_{jk}$$

$$l_{jk} = L_{jk} - \mathbf{S}_{jk}^0$$

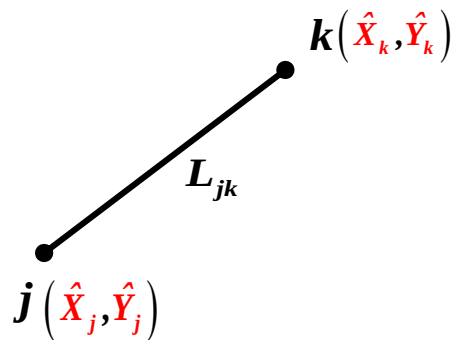


4、测边网坐标平差的误差方程

(2) 边长函数线性化变换

$$\hat{S}_{jk} = \sqrt{(\hat{X}_k - \hat{X}_j)^2 + (\hat{Y}_k - \hat{Y}_j)^2}$$

$$\hat{S}_{jk}^2 = (\hat{X}_k - \hat{X}_j)^2 + (\hat{Y}_k - \hat{Y}_j)^2$$



求全微分 $2\hat{S}_{jk}d\hat{S}_{jk} = 2(\hat{X}_k - \hat{X}_j)_0(d\hat{X}_k - d\hat{X}_j) + 2(\hat{Y}_k - \hat{Y}_j)_0(d\hat{Y}_k - d\hat{Y}_j)$

改正数替换微分

$$\begin{aligned}\hat{S}_{jk} &= \frac{(X_k^0 - X_j^0)}{S_{jk}^0}(\hat{x}_k - \hat{x}_j) + \frac{(Y_k^0 - Y_j^0)}{S_{jk}^0}(\hat{y}_k - \hat{y}_j) \\ &= -\frac{\Delta X_{jk}^0}{S_{jk}^0}\hat{x}_j - \frac{\Delta Y_{jk}^0}{S_{jk}^0}\hat{y}_j + \frac{\Delta X_{jk}^0}{S_{jk}^0}\hat{x}_k + \frac{\Delta Y_{jk}^0}{S_{jk}^0}\hat{y}_k\end{aligned}$$

$$S_{jk}^0 = \sqrt{(X_k^0 - X_j^0)^2 + (Y_k^0 - Y_j^0)^2}$$

4、测边网坐标平差的误差方程

(3) 边长改正数误差方程

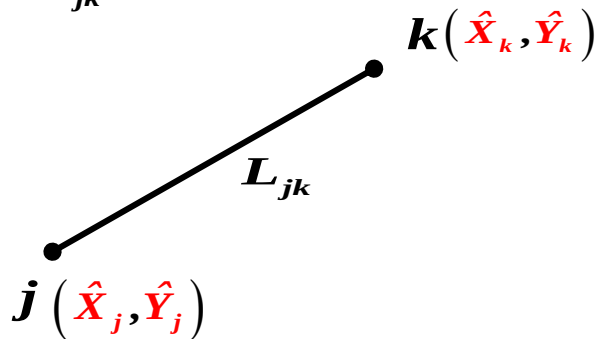
$$\hat{s}_{jk} = -\frac{\Delta X_{jk}^0}{S_{jk}^0} \hat{x}_j - \frac{\Delta Y_{jk}^0}{S_{jk}^0} \hat{y}_j + \frac{\Delta X_{jk}^0}{S_{jk}^0} \hat{x}_k + \frac{\Delta Y_{jk}^0}{S_{jk}^0} \hat{y}_k$$

用参数改正数 $\hat{s}_{jk}$ 代入边长改正数 v_{jk}

$$\begin{aligned} v_{jk} &= \hat{s}_{jk} - l_{jk} \\ &= -\frac{\Delta X_{jk}^0}{S_{jk}^0} \hat{x}_j - \frac{\Delta Y_{jk}^0}{S_{jk}^0} \hat{y}_j + \frac{\Delta X_{jk}^0}{S_{jk}^0} \hat{x}_k + \frac{\Delta Y_{jk}^0}{S_{jk}^0} \hat{y}_k - l_{jk} \end{aligned}$$

$$l_{jk} = L_{jk} - S_{jk}^0$$

$$S_{jk}^0 = \sqrt{(X_k^0 - X_j^0)^2 + (Y_k^0 - Y_j^0)^2}$$



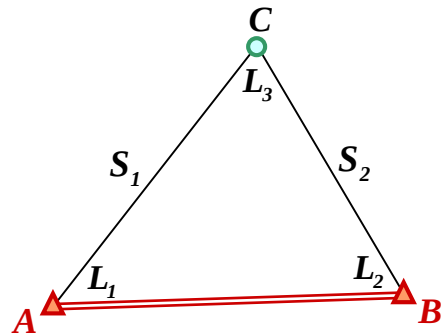
5、边角网（导线网）坐标平差的误差方程

如图边角网，观测值： L_1 、 L_2 、 L_3 、 S_1 、 S_2

平差值： $\hat{L}_1$ 、 $\hat{L}_2$ 、 $\hat{L}_3$ 、 $\hat{S}_1$ 、 $\hat{S}_2$

$n = 8$, $t = 2$,

引入参数 $\hat{X} = (\hat{X}_C \quad \hat{Y}_C)^T$



$$\alpha_{AB} = \arctan \frac{Y_B - Y_A}{X_B - X_A}$$

$$\alpha_{AC}^0 = \alpha_{AB} - L_1 \quad \alpha_{BC}^0 = \alpha_{AB} + 180 + L_2$$

$$\begin{aligned} X_C'^0 &= X_A + S_1 \cos \alpha_{AC}^0 & X_C''^0 &= X_B + S_2 \cos \alpha_{BC}^0 & X_C^0 &= (X_C'^0 + X_C''^0)/2 \\ Y_C'^0 &= Y_A + S_1 \sin \alpha_{AC}^0 & Y_C''^0 &= Y_B + S_2 \sin \alpha_{BC}^0 & Y_C^0 &= (Y_C'^0 + Y_C''^0)/2 \end{aligned}$$

角度误差方程的列立

$$\hat{L}_1 = \alpha_{AB} - \hat{\alpha}_{AC} = -\arctan \frac{\hat{Y}_C - Y_A}{\hat{X}_C - X_A} + \alpha_{AB}$$

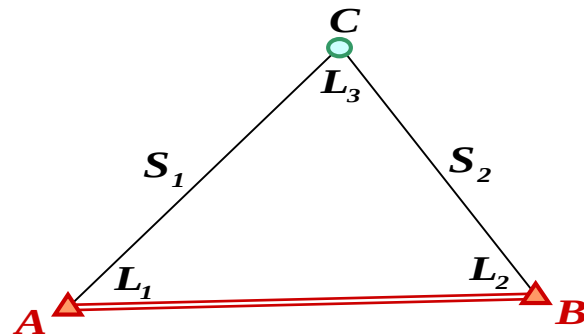
$$\hat{L}_2 = \hat{\alpha}_{BC} - \alpha_{BA} = \arctan \frac{\hat{Y}_C - Y_B}{\hat{X}_C - X_B} - (\alpha_{AB} + 180)$$

$$\hat{L}_3 = \hat{\alpha}_{CA} - \hat{\alpha}_{CB} = \arctan \frac{\hat{Y}_C - Y_A}{\hat{X}_C - X_A} - \arctan \frac{\hat{Y}_C - Y_B}{\hat{X}_C - X_B}$$

$$v_1 = -\frac{\rho'' \Delta Y_{AC}^0}{(S_{AC}^0)^2} \hat{x}_C + \frac{\rho'' \Delta X_{AC}^0}{(S_{AC}^0)^2} \hat{y}_C - l_1$$

$$v_2 = -\frac{\rho'' \Delta Y_{BC}^0}{(S_{BC}^0)^2} \hat{x}_C + \frac{\rho'' \Delta X_{BC}^0}{(S_{BC}^0)^2} \hat{y}_C - l_2$$

$$v_3 = -\left(\frac{\rho'' \Delta Y_{AC}^0}{(S_{AC}^0)^2} - \frac{\rho'' \Delta Y_{BC}^0}{(S_{BC}^0)^2} \right) \hat{x}_C + \left(\frac{\rho'' \Delta X_{AC}^0}{(S_{AC}^0)^2} - \frac{\rho'' \Delta X_{BC}^0}{(S_{BC}^0)^2} \right) \hat{y}_C - l_3$$



$$l_1 = L_1 - (\alpha_{AB} - \alpha_{AC}^0)$$

$$l_2 = L_2 - (\alpha_{BC}^0 - \alpha_{AB} - 180)$$

$$l_3 = L_3 - (\alpha_{AC}^0 - \alpha_{BC}^0)$$

边长误差方程的列立

$$\hat{S}_1 = \sqrt{(\hat{X}_C - X_A)^2 + (\hat{Y}_C - Y_A)^2}$$

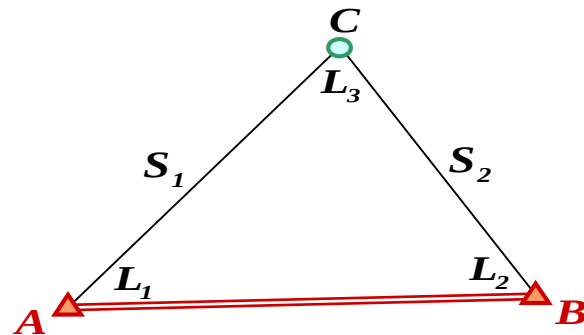
$$\hat{S}_2 = \sqrt{(\hat{X}_C - X_B)^2 + (\hat{Y}_C - Y_B)^2}$$

$$\mathbf{v}_4 = \frac{\Delta \mathbf{X}_{AC}^0}{S_{AC}^0} \hat{x}_C + \frac{\Delta \mathbf{Y}_{AC}^0}{S_{AC}^0} \hat{y}_C - l_4$$

$$\mathbf{v}_5 = \frac{\Delta \mathbf{X}_{BC}^0}{S_{BC}^0} \hat{x}_C + \frac{\Delta \mathbf{Y}_{BC}^0}{S_{BC}^0} \hat{y}_C - l_5$$

$$l_4 = \hat{S}_1 - S_{AC}^0$$

$$l_5 = \hat{S}_2 - S_{BC}^0$$



误差方程的列立

$$\mathbf{V} = \mathbf{B}\hat{\mathbf{x}} - \mathbf{l}$$

$$\hat{\mathbf{x}} = \begin{pmatrix} \hat{x}_C \\ \hat{y}_C \end{pmatrix}$$

$$\mathbf{V} = (v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5)^T$$

$$\mathbf{B} = \begin{pmatrix} \frac{\rho'' \Delta Y_{AC}^0}{(S_{AC}^0)^2} & -\frac{\rho'' \Delta X_{AC}^0}{(S_{AC}^0)^2} \\ -\frac{\rho'' \Delta Y_{BC}^0}{(S_{BC}^0)^2} & \frac{\rho'' \Delta X_{BC}^0}{(S_{BC}^0)^2} \\ -\frac{\rho'' \Delta Y_{AC}^0}{(S_{AC}^0)^2} + \frac{\rho'' \Delta Y_{BC}^0}{(S_{BC}^0)^2} & \frac{\rho'' \Delta X_{AC}^0}{(S_{AC}^0)^2} - \frac{\rho'' \Delta X_{BC}^0}{(S_{BC}^0)^2} \\ \frac{\Delta X_{AC}^0}{S_{AC}^0} & \frac{\Delta Y_{AC}^0}{S_{AC}^0} \\ \frac{\Delta X_{BC}^0}{S_{BC}^0} & \frac{\Delta Y_{BC}^0}{S_{BC}^0} \end{pmatrix}$$

$$\mathbf{l} = \begin{pmatrix} L_1 - \alpha_{AB} + \alpha_{AC}^0 \\ L_2 - \alpha_{BC}^0 + \alpha_{AB} + 180 \\ L_3 - \alpha_{AC}^0 + \alpha_{BC}^0 \\ S_1 - S_{AC}^0 \\ S_2 - S_{BC}^0 \end{pmatrix}$$

6、GPS控制网坐标平差的误差方程

在静态GPS控制测量中,在任意两个观测站上进行GPS观测, 可得到两点之间的基线向量的三维坐标差观测值。

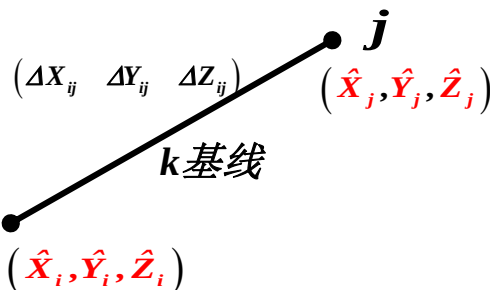
$$\begin{aligned} \text{基线k的坐标差观测值为: } & (\Delta X_{ij} \quad \Delta Y_{ij} \quad \Delta Z_{ij}) \\ \text{基线k的坐标差平差值为: } & (\Delta \hat{X}_{ij} \quad \Delta \hat{Y}_{ij} \quad \Delta \hat{Z}_{ij}) \\ \text{基线k的坐标差改正数为: } & (v_{X_{ij}} \quad v_{Y_{ij}} \quad v_{Z_{ij}}) \end{aligned} \quad \begin{pmatrix} \Delta \hat{X}_{ij} \\ \Delta \hat{Y}_{ij} \\ \Delta \hat{Z}_{ij} \end{pmatrix} = \begin{pmatrix} \Delta X_{ij} + v_{X_{ij}} \\ \Delta Y_{ij} + v_{Y_{ij}} \\ \Delta Z_{ij} + v_{Z_{ij}} \end{pmatrix}$$

引入参数为: 待定点*i,j*的坐标平差值

$$\begin{pmatrix} \hat{X}_i \\ \hat{Y}_i \\ \hat{Z}_i \end{pmatrix} = \begin{pmatrix} X_i^0 \\ Y_i^0 \\ Z_i^0 \end{pmatrix} + \begin{pmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{pmatrix} \quad \begin{pmatrix} \hat{X}_j \\ \hat{Y}_j \\ \hat{Z}_j \end{pmatrix} = \begin{pmatrix} X_j^0 \\ Y_j^0 \\ Z_j^0 \end{pmatrix} + \begin{pmatrix} \hat{x}_j \\ \hat{y}_j \\ \hat{z}_j \end{pmatrix}$$

基线k的误差方程为

$$\begin{pmatrix} \Delta \hat{X}_{ij} \\ \Delta \hat{Y}_{ij} \\ \Delta \hat{Z}_{ij} \end{pmatrix} = \begin{pmatrix} \hat{X}_j \\ \hat{Y}_j \\ \hat{Z}_j \end{pmatrix} - \begin{pmatrix} \hat{X}_i \\ \hat{Y}_i \\ \hat{Z}_i \end{pmatrix} = \begin{pmatrix} X_j^0 + \hat{x}_j \\ Y_j^0 + \hat{y}_j \\ Z_j^0 + \hat{z}_j \end{pmatrix} - \begin{pmatrix} X_i^0 + \hat{x}_i \\ Y_i^0 + \hat{y}_i \\ Z_i^0 + \hat{z}_i \end{pmatrix}$$



k基线坐标改正数的误差方程

$$\begin{pmatrix} v_{X_{ij}} \\ v_{Y_{ij}} \\ v_{Z_{ij}} \end{pmatrix} = \begin{pmatrix} \hat{x}_j \\ \hat{y}_j \\ \hat{z}_j \end{pmatrix} - \begin{pmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{pmatrix} + \begin{pmatrix} X_j^0 - X_i^0 - \Delta X_{ij} \\ Y_j^0 - Y_i^0 - \Delta Y_{ij} \\ Z_j^0 - Z_i^0 - \Delta Z_{ij} \end{pmatrix} = \begin{pmatrix} \hat{x}_j \\ \hat{y}_j \\ \hat{z}_j \end{pmatrix} - \begin{pmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{pmatrix} - \begin{pmatrix} \Delta X_{ij} - \Delta X_{ij}^0 \\ \Delta Y_{ij} - \Delta Y_{ij}^0 \\ \Delta Z_{ij} - \Delta Z_{ij}^0 \end{pmatrix}$$

矩阵形式
$$V_k = \hat{X}_{k(j)} - \hat{X}_{k(i)} - l_k$$

当网中有 p 个待定点， q 条基线向量时，观测值总数 $n=3q$ ，参数总数为 $t=3p$ ，则GPS网的误差方程式为

$$\begin{matrix} V \\ 3q,1 \end{matrix} = \begin{matrix} B \\ 3q,3p \end{matrix} \begin{matrix} \hat{X} \\ 3p,1 \end{matrix} - \begin{matrix} l \\ 3q,1 \end{matrix}$$

随机模型的一般形式为

$$D = \sigma_0^2 Q = \sigma_0^2 P^{-1}$$

k基线坐标差观测值的**方差阵**为

3个观测分量相关

$$(\Delta X_{ij} \quad \Delta Y_{ij} \quad \Delta Z_{ij})$$

$$D_{k,3,3} = \begin{pmatrix} \sigma_{\Delta X_{ij}}^2 & \sigma_{\Delta X_{ij}\Delta Y_{ij}} & \sigma_{\Delta X_{ij}\Delta Z_{ij}} \\ \sigma_{\Delta X_{ij}\Delta Y_{ij}} & \sigma_{\Delta Y_{ij}}^2 & \sigma_{\Delta Y_{ij}\Delta Z_{ij}} \\ \sigma_{\Delta X_{ij}\Delta Z_{ij}} & \sigma_{\Delta Y_{ij}\Delta Z_{ij}} & \sigma_{\Delta Z_{ij}}^2 \end{pmatrix}$$

多条基线坐标差观测值的**方差阵**为

采用单基线解求得的基线分量，不同的观测**基线**向量之间是**相互独立**的，因此多条单基线组成的GPS网，观测值方差阵**D**是分块对角矩阵。

$$D_{3n,3n} = \begin{pmatrix} D_{1,3,3} & 0 & \cdots & 0 \\ 0 & D_{2,3,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{n,3,3} \end{pmatrix}$$

小结

- 1) 引入参数有两类：未知点的高程平差值和坐标平差值
- 2) 误差方程立列有规律性，便于计算机解题
- 3) 注意改正数选取单位及方程单位统一

作业：

习题六P102 第1、3、4、6、7、11、13题

以上是本节课的内容。

同学们，下节课再见！

第六章 间接平差

§6-1 间接平差原理

§6-2 间接平差精度评价

§6-3 误差方程类型

§6-4 间接平差实例

间接平差实例-测角网坐标平差

独立同精度测得图中的6个角度
 $L_1, L_2, \dots, L_6$ 。已知点3个，未知点1个。（1）试按**间接平差法**求D点坐标平差值。（2）D点点位中误差。

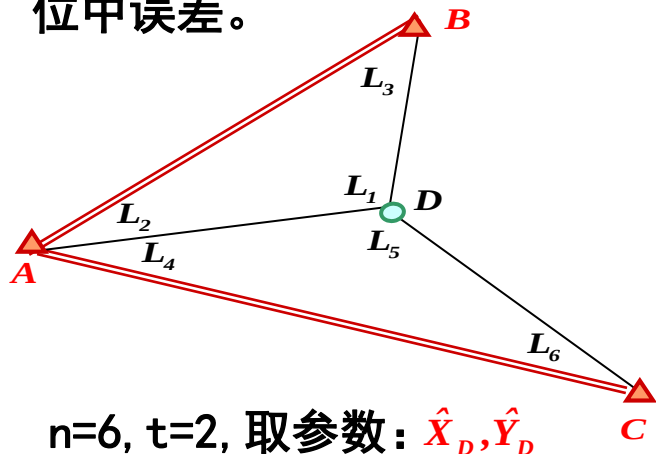


表1 角度观测值

角号	观测值 L_i	角号	观测值 L_i
1	106° 50' 42.2"	4	28° 26' 05.0"
2	30 52 44.0	5	127 48 41.2
3	42 16 39.1	6	23 45 16.2

表2 已知点数据

点名	坐标X (m)	坐标Y (m)	方位角	边长 (m)
B	13737.37	10501.92	225° 16' 38.1"	6751.24
A	8986.68	5705.03		
C	6642.27	14711.75	104 35 24.3	9306.84

解： (1)按间接平差，求D点坐标平差值

1) 计算参数近似值

先由已知点坐标计算已知方位角、已知边长；然后由已知点坐标和角度观测值计算未知点坐标近似值

$$X_D^0 = \frac{X_A \cot L_3 + X_B \cot L_2 - Y_B + Y_A}{\cot L_2 + \cot L_3} = 10122.12m$$

$$Y_D^0 = \frac{Y_A \cot L_3 + Y_B \cot L_2 + X_B - X_A}{\cot L_2 + \cot L_3} = 10312.47m$$

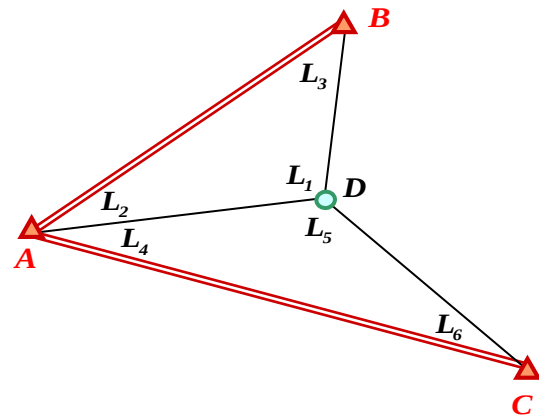


表3

再由已知点坐标和待定点坐标计算待定边坐标方位角和边长近似值。

方向	方位角近似值	边长近似值 (m)
DA	256 ° 09' 22.0"	4745
DB	2 59 59.0	3620
DC	128 20 39.0	5609

2) 列方位角改正数方程, 计算其系数

$$\delta\alpha''_{DA} = \frac{\rho'' \Delta Y_{DA}^0}{10 \cdot S_{DA}^{0^2}} \hat{x}_D - \frac{\rho'' \Delta X_{DA}^0}{10 \cdot S_{DA}^{0^2}} \hat{y}_D$$

$$\delta\alpha''_{DB} = \frac{\rho'' \Delta Y_{DB}^0}{10 \cdot S_{DB}^{0^2}} \hat{x}_D - \frac{\rho'' \Delta X_{DB}^0}{10 \cdot S_{DB}^{0^2}} \hat{y}_D$$

$$\delta\alpha''_{DC} = \frac{\rho'' \Delta Y_{DC}^0}{10 \cdot S_{DC}^{0^2}} \hat{x}_D - \frac{\rho'' \Delta X_{DC}^0}{10 \cdot S_{DC}^{0^2}} \hat{y}_D$$

边长、坐标单位为m, 坐标改正数单位为dm。系数代入改正数方程

$$\delta\alpha''_{DA} = \delta\alpha''_{AD} = -4.22 \hat{x}_D + 1.04 \hat{y}_D$$

$$\delta\alpha''_{DB} = \delta\alpha''_{BD} = 0.30 \hat{x}_D - 5.69 \hat{y}_D$$

$$\delta\alpha''_{DC} = \delta\alpha''_{CD} = 2.88 \hat{x}_D + 2.28 \hat{y}_D$$

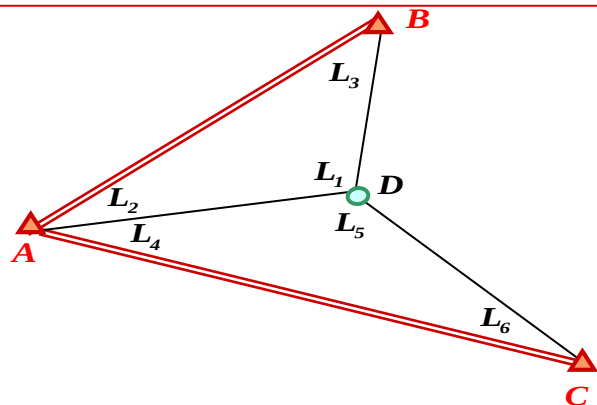


表4 系数计算

方向	ΔY^0 (m)	ΔX^0 (m)	$(S^0)^2$ (m ²)	系数 (秒 / dm)	
				$\hat{x}_D$	$\hat{y}_D$
DA	-4607	-1135	$2252 \cdot 10^4$	-4.22	1.04
DB	189	3615	$1311 \cdot 10^4$	0.30	-5.69
DC	4399	-3480	$3146 \cdot 10^4$	2.88	2.28

4) 列误差方程

$$L_1 + v_1 = \hat{\alpha}_{DB} - \hat{\alpha}_{DA}$$

$$L_2 + v_2 = \hat{\alpha}_{AD} - \hat{\alpha}_{AB}$$

$$L_3 + v_3 = \hat{\alpha}_{BA} - \hat{\alpha}_{BD}$$

$$L_4 + v_4 = \hat{\alpha}_{AC} - \hat{\alpha}_{AD}$$

$$L_5 + v_5 = \hat{\alpha}_{DA} - \hat{\alpha}_{DC}$$

$$L_6 + v_6 = \hat{\alpha}_{CD} - \hat{\alpha}_{CA}$$

$$v_1 = \delta\alpha_{DB} - \delta\alpha_{DA} - l_1 = 4.52\hat{x}_D - 6.73\hat{y}_D - 5.2$$

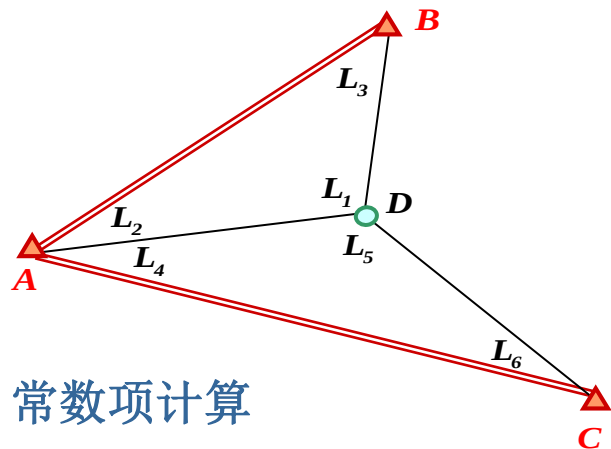
$$v_2 = \delta\alpha_{AD} - l_2 = -4.22\hat{x}_D + 1.04\hat{y}_D - 0.1$$

$$v_3 = -\delta\alpha_{BD} - l_3 = -0.30\hat{x}_D + 5.69\hat{y}_D + 0.0$$

$$v_4 = -\delta\alpha_{AD} - l_4 = 4.22\hat{x}_D - 1.04\hat{y}_D - 2.7$$

$$v_5 = \delta\alpha_{DA} - \delta\alpha_{DC} - l_5 = -7.10\hat{x}_D - 1.24\hat{y}_D + 1.8$$

$$v_6 = \delta\alpha_{CD} - l_6 = 2.88\hat{x}_D + 2.28\hat{y}_D - 1.5$$



常数项计算

$$l_1 = L_1 - (\alpha_{DB}^0 - \alpha_{DA}^0) = 5.2$$

$$l_2 = L_2 - (\alpha_{AD}^0 - \alpha_{AB}^0) = 0.1$$

$$l_3 = L_3 - (\alpha_{BA}^0 - \alpha_{BD}^0) = 0.0$$

$$l_4 = L_4 - (\alpha_{AC}^0 - \alpha_{AB}^0) = 2.7$$

$$l_5 = L_5 - (\alpha_{DA}^0 - \alpha_{DC}^0) = -1.8$$

$$l_6 = L_6 - (\alpha_{CD}^0 - \alpha_{CA}^0) = 1.5$$

5) 组成法方程，求参数及观测值的平差值

$$\begin{aligned}
 \mathbf{B} &= \begin{pmatrix} 4.52 & -6.73 \\ -4.22 & 1.04 \\ -0.30 & 5.69 \\ 4.22 & -1.04 \\ -7.10 & -1.24 \\ 2.88 & 2.28 \end{pmatrix} & \mathbf{l} &= \begin{pmatrix} 5.2 \\ 0.1 \\ 0 \\ 2.7 \\ -1.8 \\ 1.5 \end{pmatrix} \\
 N_{BB} = \mathbf{B}^T \mathbf{B} &= \begin{pmatrix} 4.52 & -4.22 & -0.30 & 4.22 & -7.10 & 2.88 \\ -6.73 & 1.04 & 5.69 & -1.04 & -1.24 & 2.28 \end{pmatrix} & \begin{pmatrix} 4.52 & -6.73 \\ -4.22 & 1.04 \\ -0.30 & 5.69 \\ 4.22 & -1.04 \\ -7.10 & -1.24 \\ 2.88 & 2.28 \end{pmatrix} &= \begin{pmatrix} 114.84 & -25.53 \\ -25.53 & 86.57 \end{pmatrix} \\
 \mathbf{W} = \mathbf{B}^T \mathbf{l} &= \begin{pmatrix} 4.52 & -4.22 & -0.30 & 4.22 & -7.10 & 2.88 \\ -6.73 & 1.04 & 5.69 & -1.04 & -1.24 & 2.28 \end{pmatrix} \begin{pmatrix} 5.2 \\ 0.1 \\ 0 \\ 2.7 \\ -1.8 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 51.58 \\ -32.05 \end{pmatrix}
 \end{aligned}$$

6) 求观测值改正数

$$\begin{pmatrix} 114.84 & -25.53 \\ -25.53 & 86.57 \end{pmatrix} \begin{pmatrix} \hat{x}_D \\ \hat{y}_D \end{pmatrix} = \begin{pmatrix} 51.58 \\ -32.05 \end{pmatrix}$$

解法方程，得

$$\begin{aligned} \hat{x}_D &= 0.393dm \\ \hat{y}_D &= -0.254dm \end{aligned}$$

求改正数

$$V = B\hat{x} - l = \begin{pmatrix} 4.52 & -6.73 \\ -4.22 & 1.04 \\ -0.30 & 5.69 \\ 4.22 & -1.04 \\ -7.10 & -1.24 \\ 2.88 & 2.28 \end{pmatrix} \begin{pmatrix} 0.393 \\ -0.254 \end{pmatrix} - \begin{pmatrix} 5.2 \\ 0.1 \\ 0 \\ 2.7 \\ -1.8 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -1.7'' \\ -2.0 \\ -1.6 \\ -0.8 \\ -0.7 \\ -0.9 \end{pmatrix}$$

求观测值平差值

$$\hat{L} = L + V = \begin{pmatrix} 105^{\circ} 50' 40.5'' \\ 30^{\circ} 52' 42.0'' \\ 42^{\circ} 16' 37.5'' \\ 28^{\circ} 26' 04.2'' \\ 127^{\circ} 48' 40.5'' \\ 123^{\circ} 45' 15.3'' \end{pmatrix}$$

求D点坐标平差值

$$\hat{X}_D = X_D^0 + \hat{x}_D = 10122.16m$$

$$\hat{Y}_D = Y_D^0 + \hat{y}_D = 10312.44m$$

(2) 求D点点位中误差

$$\hat{\sigma}_0^2 = \frac{V^T P V}{r} = \frac{1}{r} \sum_{i=1}^n p_i v_i^2 = \frac{1}{4} (1.7^2 + 2.0^2 + 1.6^2 + 0.8^2 + 0.7^2 + 0.9^2) = \frac{1}{4} \times 11.39 = 2.84 (")^2$$

$$\hat{\sigma}_0 = 1.7'' \quad N_{BB} = \begin{pmatrix} 114.84 & -25.53 \\ -25.53 & 86.57 \end{pmatrix}$$

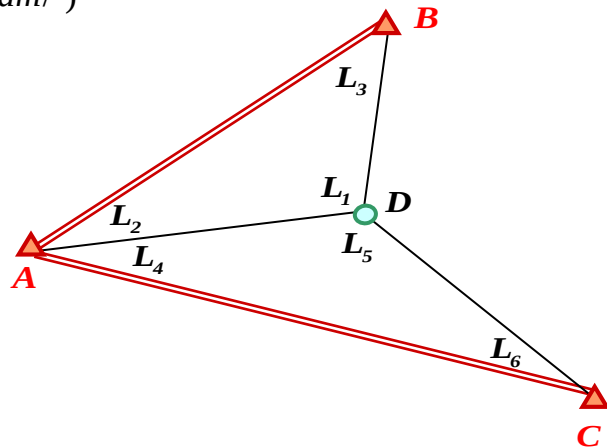
$$Q_{\hat{X}\hat{X}} = N_{BB}^{-1} = \frac{1}{9289.92} \begin{pmatrix} 86.57 & 25.53 \\ 25.53 & 114.84 \end{pmatrix} = \begin{pmatrix} 9.32 & 2.75 \\ 2.75 & 12.36 \end{pmatrix} \times 10^{-3} (dm/")^2$$

$$\hat{\sigma}_{\hat{X}_D}^2 = \hat{\sigma}_0^2 Q_{\hat{X}_D \hat{X}_D} = 2.84 \times 9.32 \times 10^{-3} = 26.47 \times 10^{-3} (dm^2)$$

$$\hat{\sigma}_{\hat{Y}_D}^2 = \hat{\sigma}_0^2 Q_{\hat{Y}_D \hat{Y}_D} = 2.84 \times 12.36 \times 10^{-3} = 35.10 \times 10^{-3} (dm^2)$$

$$\hat{\sigma}_D^2 = \hat{\sigma}_{\hat{X}_D}^2 + \hat{\sigma}_{\hat{Y}_D}^2 = 61.57 \times 10^{-3} (dm^2)$$

$$\hat{\sigma}_D = 2.5 cm$$



以上是本节课的内容。

同学们，下节课再见！

第八章 概括平差

§8-1 概括平差

§8-2 基本平差方法总结

1、四种基本平差方法的总表1(数学模型)

方法平差	引入参数 $\hat{X}$	$\hat{L}, \hat{X}$ 的函数模型			$V, \hat{x}$ 的函数模型	随机模型 取 σ_0^2
		名称	非线性	线性		
条件平差	无参	条件方程	$F_{r1}(\hat{L}_{n1}) = 0$	$A_{rn} \hat{L}_{n1} + A_{r1} A_0 = 0$	$A_{rn} V_{n1} + W_{r1} = 0$ $W_{r1} = F(L) = AL + A_0$	$D_{nn} = \sigma_0^2 Q = \sigma_0^2 P_{nn}^{-1}$
附参条件平差	u<t独参	附参条件方程	$F_{c1}(\hat{L}_{n1}, \hat{X}_{u1}) = 0$	$A_{cn} \hat{L}_{n1} + B_{cu} \hat{X}_{u1} + A_{c1} A_0 = 0$	$A_{cn} V_{n1} + B_{cu} \hat{X}_{u1} + W_{c1} = 0$ $W_{c1} = F(L, X^0) = AL + BX^0 + A_0$	$D_{nn} = \sigma_0^2 Q = \sigma_0^2 P_{nn}^{-1}$
间接平差	t独参	误差方程	$\hat{L}_{n1} = F_{n1}(\hat{X}_{t1})$	$\hat{L}_{n1} = B_{nt} \hat{X}_{t1} + d_{n1}$	$V_{n1} = B_{nt} \hat{X}_{t1} - l_{n1}$ $l = L - F(X^0) = L - (BX^0 + d)$	$D_{nn} = \sigma_0^2 Q = \sigma_0^2 P_{nn}^{-1}$
附限间接平差	t+s参 t独s限	误差方程 +限制条件方程	$\hat{L}_{n1} = F_{n1}(\hat{X}_{u1})$ $\Phi_{s1}(\hat{X}_{u1}) = 0$	$\hat{L}_{n1} = B_{nu} \hat{X}_{u1} + d_{n1}$ $C_{su} \hat{X}_{u1} - C_{s1} X^0 + W_{s1} = 0$	$V_{n1} = B_{nu} \hat{X}_{u1} - l_{n1}$ $C_{su} \hat{X}_{u1} + W_{s1}^x = 0$ $W_{s1}^x = \Phi_s(X^0)$	$D_{nn} = \sigma_0^2 Q = \sigma_0^2 P_{nn}^{-1}$

$$\hat{L}_{n1} = L_{n1} + V_{n1}$$

$$\hat{X}_{u1} = \hat{X}_{u1}^0 + \hat{x}_{u1}$$

$$A = \left(\frac{\partial F}{\partial \hat{L}} \right)_0$$

$$B = \left(\frac{\partial F}{\partial \hat{X}} \right)_0$$

$$C = \left(\frac{\partial \Phi_s}{\partial \hat{X}} \right)_0$$

1、四种基本平差方法的总表2(平差结果)

方法平差	引入参数 $\hat{x}$	$V, \hat{x}$ 函数模型	平差原则 目标函数	法方程	$V, \hat{x}$ 求解	协因数阵 $Q_{\hat{L}\hat{L}} = Q - Q_{VV}$
条件平差	无参	$AV + W = 0$ $rn \quad n1 \quad r1$	$\Phi(V) = \frac{V^T P V}{11}$ $- 2 \frac{K^T (AV + W)}{1r} = \min$	$N_{aa} K + W = 0$ $rr \quad r1$ $W = F(L) = AL + A_0$	$V = -QA^T N_{aa}^{-1} W$ $n1$	$Q_{VV} = QA^T N_{aa}^{-1} AQ$
附参条件平差	$u < t$ 独参	$AV + B\hat{x} + W = 0$ $cn \quad n1 \quad cu \quad u1 \quad c1$	$\Phi(V, \hat{x}) = \frac{V^T P V}{11}$ $- 2 \frac{K^T (AV + B\hat{x} + W)}{1c} = \min$	$\begin{cases} N_{aa} K + B\hat{x} + W = 0 \\ B^T K = 0 \end{cases}$ $cc \quad c1 \quad cu \quad u1 \quad c1$ $W = F(L) = AL + A_0$	$\hat{x} = -N_{bb}^{-1} B^T N_{aa}^{-1} W$ $K = -N_{aa}^{-1} (B\hat{x} + W)$ $V = QA^T K$	$Q_{KK} = N_{aa}^{-1} - N_{aa}^{-1} B Q_{\hat{x}\hat{x}} B^T N_{aa}^{-1}$ $Q_{VV} = QA^T Q_{KK} AQ$ $Q_{\hat{x}\hat{x}} = N_{bb}^{-1}$
间接平差	t 独参	$V = B\hat{x} - l$ $n1 \quad nt \quad t1 \quad n1$	$\Phi(\hat{x}) = \frac{V^T P V}{11} = \min$	$N_{BB} \hat{x} - W = 0$ $tt \quad t1 \quad t1$ $W = B^T Pl$	$\hat{x} = N_{BB}^{-1} W$ $V = B\hat{x} - l$ $n1 \quad nt \quad t1 \quad n1$	$Q_{VV} = Q - BN_{BB}^{-1} B^T$ $Q_{\hat{x}\hat{x}} = Q_{\hat{x}\hat{x}} = N_{BB}^{-1}$
附限间接平差	$t+s$ 参 t 独 s 限	$V = B\hat{x} - l$ $n1 \quad nu \quad u1 \quad n1$ $C\hat{x} + W_x = 0$ $su \quad u1 \quad s1$	$\Phi(\hat{x}) = \frac{V^T P V}{11}$ $+ 2 K_s^T (C\hat{x} + W_x) = \min$ $1s \quad s1$	$N_{BB} \hat{x} + C_s^T K_s - W = 0$ $uu \quad u1 \quad us \quad s1 \quad u1$ $C\hat{x} + W_x = 0$ $su \quad u1 \quad s1$ $W = B^T Pl$	$K_s = N_{CC}^{-1} (CN_{BB}^{-1} W + W_x)$ $\hat{x} = -N_{BB}^{-1} (C^T K_s - W)$ $V = B\hat{x} - l$ $n1 \quad nu \quad u1 \quad n1$	$Q_{\hat{x}\hat{x}} = N_{BB}^{-1} - N_{BB}^{-1} C^T N_{CC}^{-1} CN_{BB}^{-1}$ $Q_{VV} = Q - BQ_{\hat{x}\hat{x}} B^T$

$$\hat{L}_{n1} = L_{n1} + V_{n1}$$

$$\hat{x}_{u1} = \hat{x}_{u1}^0 + \hat{x}_{u1}$$

$$N_{aa} = AQA^T$$

$$N_{bb} = B_{uc}^T N_{aa}^{-1} B_{cu}$$

$$N_{BB} = B^T PB$$

$$N_{CC} = C N_{BB}^{-1} C^T$$

2、四种基本平差方法异同点

不同点:

- 1) 平差方法引入参数的背景不同
- 2) 平差的函数模型个数、形式不同

相同点:

- 1) 平差的随机模型相同
- 2) 平差原则相同-最小二乘原则
- 3) 同一平差问题采用不同平差方法时, 平差结果相同。

条件平差于间接平差中, $AB=0$

$$V = QA^T K \quad \text{(条件平差)}$$

$$B^T P V = 0 \quad \text{(间接平差)}$$

$$B^T P Q A^T K = B^T A^T K = (AB)^T K = 0$$

$$K \neq 0$$

$$AB = 0$$

$m \quad nt$

3、平差方法的选择

1) 一般情况下，采用间接平差或条件平差。

r 较大时， N_{aa} 阶数大，宜用间接平差

t 较大时， N_{BB} 阶数大，宜用条件平差

2) 已知条件或观测条件特殊，则可选附有参数的条件平差，
或附有限制条件的间接平差

3) 工程上间接平差较常用，其误差方程规律性强，便于立列
和编程解算。

作业

习题八：1、5、8、9

以上是本节课的内容。

同学们，下节课再见！
